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τὰ τοιαῦτα τοῖς μαθηταῖς ἐπὶ σχολῆς φράζουσιν,
οὓς ἂν βούλωνται ὁμοίους αὑτοῖς ποιῆσαι.

PLATO.

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I.—ON THE ELEMENTARY PRINCIPLES OF THE APPLICATION OF ALGEBRAICAL SYMBOLS TO GEOMETRY.

By D. F. GREGORY, B.A. Trin. Coll.

IN several previous papers in this Journal, I have considered the principles on which certain symbols of operation become subject to the same rules of combination as the symbols of number, which are those usually handled in Algebra. The general theory of this subject I gave in a paper (to which I have elsewhere referred) on the Nature of Symbolical Algebra; in which I endeavoured to exhibit distinctly the principles on which various branches of science may be symbolized—that is to say, on which their study is facilitated by expressing the operations by means of symbols. I use the word *operation* for the purpose of avoiding anything like limitation in the subjects which the symbols may represent, as is too apt to be the case when we employ the word *quantity*, which is generally made to be synonymous with number. Among the sciences whose symbolization I there considered, that of Geometry is the most important; and on that account I wish here to treat of it more at large, especially because it appears to me that the theory of the representation of geometrical quantities by numerical symbols is usually but little attended to, and some obscurity still hangs over the subject. In treating of this matter, I may perhaps appear to some to be raising difficulties where there are none; but I think that a little consideration will show to these persons that the question is not quite so simple as might at first be imagined. Much attention has been bestowed on the theory of the representation of

direction by means of the symbols $+$ and $-$, but the principles on which lines, areas, and solids are represented by numbers has been but little discussed. It is to the latter of these subjects that my remarks will be first directed, and I shall afterwards develop my views of the former.

In the paper I have referred to, I lay down the principle, that an algebraical symbol can only represent an operation in any other science when it is subject to the same laws of combination as that operation. In fact, that as Algebra takes cognizance only of the laws of combination of the symbols, and not of their meaning—in the eye of that science the symbol and the operation are identical. When we turn to the interpretation of our results, we must of course consider the meanings of the symbols—but such interpretation is out of the province of Algebra, and belongs to the science, the operations of which are symbolized. Now, in applying these principles to Geometry, we have first to become acquainted with the operations which require to be symbolized, and then to consider the laws of combination to which they are subject, in order that we may know under which family of algebraical symbols they are to be classed. The ideas with which we are concerned in Geometry are those of magnitude and direction. The former is of three kinds—linear, plane, and solid; and the question is, of what sort of operations these may be considered as the result. Such a one I conceive to be *transference* in *one direction;* for by proper combinations of operations of this description we can represent magnitudes of all kinds. Some persons may think it strange to introduce such an idea as that of transference into so simple a subject as Geometry; but in defence of its adoption, I think it only necessary to plead the simplicity and uniformity of the explanations it affords of the principle of the application of Algebra to Geometry: and I may add, that we are not here considering how Geometry may be treated *geometrically*, but *symbolically;* and we must be content to do so in the way which the subject most readily permits. Besides, for my own part, I think that the idea of transference is quite as simple and elementary as any which occurs in Geometry, and offers itself as readily to the mind of the student. Having fixed on an operation which is to be symbolized, we have also to consider what may be the subject of that operation. The simplest geometrical idea, and that which suits our purpose, is the idea of a point. We may, if we choose, represent this by a symbol, as we represent the fundamental subject-idea in Arithmetic by the symbol 1: but this is not necessary; for, as in Algebra, we have only to consider the combinations of symbols of operation—the subject, being always the same, may be understood, and the symbol for it omitted. Thus it is that we omit in Arithmetic the symbol for unity, which nevertheless requires to be understood at every step as the subject, without which the whole would be unintelligible.

Now, let us assume a to be a symbol representing transference in *one constant* direction through a given space; then, representing

the subject-point by the symbol (.), the compound symbol

$$a\,(.)$$

will represent a straight line, as the result of transferring a point through a given space in a constant direction. But as we have agreed to omit the subject-symbol, a line of a given length will be simply represented by the symbol a, which now does not represent the operation, but the result of the operation on the subject.

Again, we may combine this symbol with another symbol for transference in some other given direction, and we may ask the meaning of such a combination as

$$b\,\{a\,(.)\},$$

or, omitting the symbol for the subject,

$$b\,(a).$$

This, it is clear, must signify the transference of a line in one constant direction, that is, the line must move parallel to itself, by which means it will trace out a parallelogram, whose sides are represented by a and b.

In the same way in which we have combined two symbols of transference we may combine three, and ask the meaning of the expression $c\{a\,(b)\}$. This will, on the same principle, represent the transference of a plane in one constant direction, that is, the transference of a plane parallel to itself, the result of which is a parallelopiped. If we combine more symbols than these, we find no geometrical interpretation for the result. In fact, it may be looked on as an impossible geometrical operation; just as $\sqrt{-1}$ is an impossible arithmetical one. For a solid, having equal relations to the three dimensions of space, cannot have any relation with one particular direction, which refers only to one dimension, and direction is essentially involved in the operation we have been considering.

From what has preceded, it appears that we are able, by the combination of the symbol of one kind of operation, to represent the three different geometrical magnitudes—lines, areas, and solids; but, as yet, nothing has been said to point out the *algebraical* nature of these symbols, so that we cannot tell whether or not they coincide algebraically with the symbols for numbers. So far as we have gone, we have not shown how the study of Geometry may be facilitated by having its operations symbolized, as we know not how to treat the symbols, some combinations of which we have been interpreting. But we shall now proceed to show, that these symbols are subject to the two laws of combination which characterize the symbols of number, the ordinary subjects of algebraical operations, viz. the commutative law and the distributive law.

We have found that $b\,(a)$ represents a parallelogram, the sides of which are a and b; and in the same way $a\,(b)$ must represent a parallelogram whose sides are also a and b, and which is identical

with the former, as the relative inclination of the sides is the same. Hence it follows, that when a and b represent the geometrical operation of transference in a given direction,

$$a(b) = b(a),$$

or the symbols are *commutative.*

Again, with respect to the distributive law: supposing that the symbol $+$ represents the simple arithmetical idea of addition, (the reason for which restriction will be seen afterwards,) $a+b$ will represent a line resulting from the transference of a point in the same direction through distances a and b, and $c(a+b)$ will represent a parallelogram whose sides are c and $a+b$. But $c(a)$ and $c(b)$ will represent respectively parallelograms, whose sides are c, a and c, b, so that $c(a)+c(b)$ will represent the sum of these parallelograms. But, by the first proposition of the second book of Euclid, we know that the sum of these is equal to the first parallelogram. It is true that the proposition in Euclid is proved only for rectangles, but the principle of the demonstration applies to all parallelograms whatsoever. From this it follows, that when c, a, b represent the geometrical idea of transference in a given direction,

$$c(a+b) = c(a) + c(b),$$

or the symbols are distributive.

We are now enabled to see why we can represent geometrical ideas by arithmetical symbols, so as to render geometrical research easier from our previous acquaintance with arithmetical combinations. It is because the symbols in both cases are subject to the same laws of combination, and therefore in the eye of Algebra are identical, at least so far as these laws (which are the algebraical definitions) are concerned. Whatever, therefore, may have been proved in Arithmetic, in dependence solely on these laws, is equally true in Geometry, provided always that we can interpret the result; for there is no reason why we should always be able to interpret a symbolical result either geometrically or arithmetically. And indeed, in Geometry the uninterpretability is soon presented to us in the combination of more than three symbols of transference. From this it appears why areas and solids may be represented by the product of the symbols of lines, or rather by the apparent product: for when a and b are geometrical symbols, we cannot talk of their being multiplied together—but we see that the operation of one on the other bears a close resemblance to the arithmetical operation of multiplication, and from the identity of the laws of combination they may be considered algebraically as the same, though the meanings be wholly different. This question as to the possibility of representing areas and solids by means of the apparent multiplication of the symbols for lines, has always appeared to me to be one of great difficulty in the application of Algebra to Geometry: nor has the difficulty, I think, been properly met in works on the subject. It is not sufficient to say, as is usually done, that if we divide each of

the lines into a certain number of units, the number of superficial units in the parallelogram will be equal to the product of the number of units in the two lines; it is also necessary to show how a superficial unit can be represented by the product of two linear units, and this I think cannot be done except on the principle which has here been used.

It is to be observed, that in all which has preceded we have supposed the symbols to represent transference in a constant direction. This limitation is necessary in defining our symbols; for if we were to suppose the direction to vary during the progress of the transference, the same laws would not be found to hold with respect to these symbols as we have seen to hold for the symbols we considered, and we should then be unable to reduce geometrical investigations to processes of arithmetical calculation. We might, certainly, if we chose, use symbols representing different kinds of transference, and we might employ ourselves in investigating their nature and the laws of their combination; but having done so, we should derive no assistance from any previous labours in the science of symbols. It is solely from the previous knowledge which we have of the combinations of arithmetical symbols, that we are enabled to facilitate our researches by the application of Algebra to Geometry, or to any science whatever. And thus it is, that any improvement or discovery in Algebra, however isolated and useless it at first appear, may become ultimately of the utmost importance for the prosecution of other branches of knowledge.

Hitherto we have confined ourselves to the consideration of the means of representing symbolically the geometrical ideas of magnitude; and we have shown how the combination of these symbols to represent areas and solids, bears an analogy to the processes of multiplication in Arithmetic: we shall now proceed to consider the symbolization of direction, and to show that the symbols we adopt bear a striking analogy to a well-known arithmetical symbol.

Direction, in ordinary Plane Geometry, is estimated by means of rectilinear angles, which affords us an easy means of symbolizing this geometrical idea; for by supposing a straight line to revolve round a point situate within it, we can make it generate any given angle. This, therefore, is the operation which we shall express by a symbol, and the laws of which we are to investigate. It is clear, in the first place, that if we take some standard angle as that which is to be the result of the operation symbolized, we may produce multiples or submultiples of that angle by performing the operation a certain number of times, or by performing a certain part of the operation. It is therefore necessary to choose some angle for our standard, and the most convenient for our purpose is that produced by a complete revolution of the line, or revolution through four right angles. Let us assume, then, the symbol Λ to represent the operation of making a line revolve through four

right angles, so that, a representing a line in a given direction, $\Lambda(a)$, will represent the same line inclined at an angle equal to four right angles,—that is to say, in a direction coinciding with the original direction. If we repeat the operation, $\Lambda\{\Lambda(a)\}$, or in accordance with ordinary algebraical notation $\Lambda^2(a)$, will represent a line inclined to the original at an angle equal to eight right angles, and so on for any number of times that the operation may be performed. As we have introduced integer indices attached to the operation Λ, we may also use fractional indices, and enquire what is the meaning of such an expression as $\Lambda^{\frac{1}{2}}(a)$ or $\Lambda^{\frac{1}{3}}(a)$. In accordance with the algebraical laws for the combination of indices, we easily see that $\Lambda^{\frac{1}{2}}$ must signify an operation which, being performed twice, will give birth to Λ. Such will be the turning of a line through two right angles, or 180^0, so that $\Lambda^{\frac{1}{2}}(a)$ will represent a line measured in the opposite direction from the original line. In the same way $\Lambda^{\frac{1}{3}}$ must signify the turning of a line through one-third of four right angles, or 120^0, as that operation being performed thrice will be equivalent to the turning of a line through four right angles. And generally $\Lambda^{\frac{1}{n}}$ will signify the turning of a line through the n^{th} part of four right angles, or $\dfrac{360^0}{n}$. Thus, by the use of the simple algebraical notation of indices, joined to the geometrical operation of turning a line through a given angle, we are able to express the operation of turning a line through any angle whatsoever, and so to express all relations of directions between lines situate in a plane. It is to be observed, that since the operation of turning a line through four right angles, or through any multiple of four right angles, brings it back to its original position, the effect of any number of repetitions of the operation Λ is the same, which may be expressed algebraically by saying that

$$\Lambda^n = \Lambda,$$

n being any integer, which is a law of combination of Λ, and may be considered as its algebraical definition. Now, this is the very law which is known to belong to the arithmetical operation of addition usually represented by $+$, since we have then

$$+\,+ = +, \text{ and therefore } +^n = +,$$

n being any integer. Hence it appears, that as the arithmetical operation of addition, and the geometrical operation of turning a line through four right angles, are subject to the same law of combination, they are, so far as that is concerned, algebraically identical, and may be represented by the same symbol. Such, indeed, has long been the case, for the arithmetical symbols for addition and subtraction, along with certain modifications of them, are constantly

used to represent geometrical direction. This has given rise to much difficulty and many attempts at explanation; some persons wishing to show that the geometrical operation might be supposed to be derived from the arithmetical, but not finding it very easy to do so in a satisfactory manner—others being inclined to found their views of some points in the arithmetical theory on the basis of the geometrical idea, interpreting the former by the latter. I believe, that the more closely the subject is examined, the more clearly it will be seen, that there is really no resemblance in *kind* between the two operations, but only an identity in the laws of combination; and if this be kept steadily in view, all the difficulties which have been observed in this part of mathematics, and on which so much has been written, will receive a satisfactory explanation. This double meaning of $+$ is the reason of the limitation to the meaning of that symbol assumed in p. 4.

We have only considered the operation Λ or $+$, as we may now term it, in connection with the symbol for a line, as it was with reference to the direction of a line that its definition was made. But this symbol may also receive interpretation in another case, to which its original definition does not directly refer. It is not *necessary* that it should admit of any other geometrical interpretation, but such is found to be the case when it is applied to areas. The position of a line is determined by the direction in which its length lies; but the position of a plane cannot be determined in like manner by its extension, since that has two dimensions, and direction has only one. But the position of an area may be determined by the direction of the face of the plane, which can be referred to that of any straight line inclined to it at a given angle (such as a right angle), so that we know how one plane is related to another if we know in what direction the face of each is presented. Now, supposing an area to revolve round any line in its own plane, we can make it assume any position we please; and it is easy to see that the operation of turning the area completely round is subject to the same law as that of turning the line, that is to say, that when it is repeated any number of times the result is the same, since the area will always present the same face. Hence it follows, that these two operations may be represented by the same symbol; so that if in any process of Analytical Geometry we find the symbol $+$, which was originally applied to the symbol for a line, ultimately applied to the symbol for an area, we are able to interpret it. This view of the meaning of $+$, when applied to the symbol for an area, enables us to offer an explanation of a difficulty in Analytical Geometry.

If x, Ay (fig. 1.) be a system of rectangular coordinates, we know, from what has been previously said concerning the representation of the direction of lines, that any abscissa measured along x will be affected with $+$, and any abscissa measured along x' will be affected with $+^{\frac{1}{2}}$ or $-$; and similarly, any

ordinate measured along Ay will be affected with $+$, and any along Ay' with $+^{\frac{1}{2}}$ or $-$. Therefore the coordinates of a point P will be

$$+x, +y, \quad -x, +y, \quad -x, -y, \quad +x, -y,$$

according as it is in the first, second, third, or fourth quadrant. Now, the rectangle Ax Py being represented by the product of the symbols representing its sides, will be

$$+xy, \quad -xy, \quad +xy, \quad -xy,$$

according as it is in the first, second, third, or fourth quadrant. The question then is, what meaning we are to attach to these expressions. It will be seen by a glance at the figure, that if the rectangle AxP$_1y$ turn round the line Ay, or the line Ax through half a circumference, it will occupy the place of Ax'P$_2y$, or Ay'P$_4x$, and therefore these rectangles may be considered as resulting from the turning of the original rectangle round Ay or Ax through half a circumference, so as to present the other face of the plane. Now, we have just shown that the operation of turning a plane through a complete circumference, so as to present the same face as before, may be represented by $+$, and therefore the operation of turning it through half a circumference may be represented by $-$. Therefore the negative signs attached to expressions for the rectangles in the second and fourth quadrants, are to be interpreted as signifying that these rectangles are equivalent to the original rectangles turned through half a circumference round Ax or Ay, just as the line Ax' would be produced by turning Ax through half a circumference. With respect to the rectangle in the third quadrant, which forms the chief point of difficulty, it can be derived either from that in the second or that in the fourth, by turning them through half a circumference round Ax' or Ay'. And as both of these rectangles present the face opposite to that of the primary rectangle, it is quite consistent with, and indeed follows from the definition of $+$, that the rectangle in the third segment should be represented by $+xy$, since it is derived from the primary rectangle by that rectangle being turned through a circumference, so that it presents the same face in the same direction as it did at first. Or if we suppose that the area in the second or fourth quadrant, instead of continuing to revolve in the same direction as that by revolving in which it was derived from the area in the first quadrant, revolves back so as to undo the operation previously performed, the same result will follow. For the area in the first quadrant being represented by $+xy$, that in the second, being the former turned round Ay through half a circumference, will be represented by $+^{\frac{1}{2}}+xy$: while the area in the third quadrant, being derived from that in the second by its being turned round Ax' through half a circumference in the opposite direction, will be represented by $+^{-\frac{1}{2}}+^{\frac{1}{2}}+xy$, or $+xy$, as in the first

quadrant, which ought to be the case, as the same face as before is presented. The same result of course will follow, if we consider the area in the third quadrant as derived from that in the fourth.

These explanations of the meanings of the symbols + and −, when applied to areas, are consistent with the original definition, and are closely analogous to their significations when applied to lines, so that I think they must be deemed satisfactory. Should it now be asked whether these principles can be applied to solids, so as to explain the meaning of the symbols + and − prefixed to those of parallelopipeds, I have to answer that they do not; and the reason I conceive to be, as I said before on another subject, that a solid being extended in three dimensions has no relation to one direction, which is essentially only of one dimension. A face or an edge of the solid may be referred to one direction, but the solid itself cannot be so referred. Such expressions as $+abc$ or $-abc$ are, I hold, uninterpretable consistently with the geometrical meaning we attach to the symbols + and −. By calling them uninterpretable, I put them in the same class in Geometry as the symbol $\sqrt{-1}$ in Arithmetic; we do not at present see any interpretation for them, though there is no reason why farther progress and more extended views in Arithmetic and Geometry should not enable us to understand what is at present beyond our comprehension.

II.—ON THE TRANSFORMATION OF A CERTAIN ANALYTICAL EXPRESSION.*

The following very useful formula, mentioned by the Author of a Paper in the fourth Number, viz.

$$(cy-bz)^2+(az-cx)^2+(bx-ay)^2$$
$$=(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2,$$

may be considered as a particular case of a more general one; to wit,

$$(m_0n_1-m_1n_0)(m_2n_3-m_3n_2)+(n_0l_1-n_1l_0)(n_2l_3-n_3l_2)$$
$$+(l_0m_1-l_1m_0)(l_2m_3-l_3m_2)$$
$$=(l_0l_2+m_0m_2+n_0n_2)(l_1l_3+m_1m_3+n_1n_3)$$
$$-(l_1l_2+m_1m_2+n_1n_2)(l_0l_3+m_0m_3+n_0n_3),$$

which is easily verified, and may be remembered by observing, that the combinations of indices of the same letter which occur in the positive part of the second member of the equation, are the

* From a Correspondent.

same as would be found in any positive term of the developement of the first number, and similarly for the negative part. From this formula may be immediately deduced some propositions in Spherical Trigonometry. For let there be any three rectangular axes, and from the origin let any four straight lines be drawn, denoted by the figures 0, 1, 2, 3 respectively; and let $l_0 m_0 n_0$, $l_1 m_1 n_1$, $l_2 m_2 n_2$, $l_3 m_3 n_3$, be the cosines of the angles which they make with the positive semiaxes. Also, let six planes be drawn, each containing two of these lines: and let the angle between any two of the lines, 1 and 2 for example, be denoted by the symbol (1.2); and the angle between the plane containing the lines 1, 2, and that containing 0, 3, by the symbol $\binom{1.2}{0.3}$. Then we have the following equations, viz.

$$\cos(0.1) = l_0 l_1 + m_0 m_1 + n_0 n_1,$$
$$\cos(2.3) = l_2 l_3 + m_2 m_3 + n_2 n_3,$$
&c.

Moreover, it is easily seen that the cosines of the angles which a perpendicular to the plane 0, 1 makes with the axes, will be the quantities $m_0 n_1 - m_1 n_0$, $n_0 l_1 - n_1 l_0$, $l_0 m_1 - l_1 m_0$, divided respectively by the square root of the sum of their squares, that is, by

$$\{1 - (l_0 l_1 + m_0 m_1 + n_0 n_1)^2\}^{\frac{1}{2}},$$

and so for any other plane.

Hence it is evident, that for the angle between any two planes, for instance, 0, 1 and 2, 3, we shall have

$$\cos\binom{0.1}{2.3} = (m_0 n_1 - m_1 n_0)(m_2 n_3 - m_3 n_2)$$
$$+ (n_0 l_1 - n_1 l_0)(n_2 l_3 - n_3 l_2) + (l_0 m_1 - l_1 m_0)(l_2 m_3 - l_3 m_2)$$

divided by

$$\pm \sqrt{\{1 - (l_0 l_1 + m_0 m_1 + n_0 n_1)^2\}\{1 - (l_2 l_3 + m_2 m_3 + n_2 n_3)^2\}};$$

or, substituting for the numerator its value as given by the formula at the beginning of this article,

$$\cos\binom{0.1}{2.3} = \pm \frac{\cos(0.2)\cos(1.3) - \cos(0.3)\cos(1.2)}{\sin(0.1).\sin(2.3)}.$$

If we suppose a sphere described about the origin as centre, the four lines we have been considering will meet its surface in four points, which will be the angular points of a quadrilateral figure, whose sides and diagonals will be the intersections of the sphere with the six planes above mentioned. And the formula just written evidently gives the angle between any two opposite sides, or between the two diagonals, in terms of the sides and diagonals, according as we adopt different arrangements of the figures 0, 1, 2, 3.

Thus, let *a*, *b*, *c*, *d* be the sides, and δ, δ' the diagonals. And

suppose a is opposite to c. Then, if ϕ is the angle between the diagonals, we have

$$\cos\phi = \pm \frac{\cos a \,.\, \cos c - \cos b \,.\, \cos d}{\sin\delta \,.\, \sin\delta'},$$

and calling θ the angle between a and c,

$$\cos\theta = \pm \frac{\cos b \cos d - \cos\delta \cos\delta'}{\sin a \,.\, \sin c}.$$

If in this last equation we suppose $d = 0$, the figure becomes a triangle, and the diagonals coincide with the sides a and c respectively; and we get immediately the common formula for the angle opposite the side b, viz.

$$\cos\beta = \frac{\cos b - \cos a \cos c}{\sin a \sin c}.$$

If in the former equation we suppose the diagonals at right angles to one another, we must have $\cos\varphi = 0$, and therefore

$$\cos a \cos c = \cos b \cos d.$$

This will include the case of a triangle with a perpendicular drawn from any angle to the opposite sides, if we suppose three of the angular points of the quadrilateral to be in the same great circle. In this case, let a and b be two sides, and α, β the two parts into which the third side is divided by the point where it is met by a perpendicular from the opposite angle, and the equation last written gives evidently

$$\cos a \cos\beta = \cos b \cos\alpha,$$

α being supposed contiguous to a, and β to b.

M. N. N.

III.—ON THE NUMBER OF NORMALS THAT CAN BE DRAWN FROM A GIVEN POINT TO AN ALGEBRAICAL SURFACE: BY M. TERQUEM.

[From Liouville's *Journal de Mathématiques*, vol. IV. p. 175.]

Theorem. THE number of normals which can be drawn from a given point to an algebraical surface of the degree m, is equal to $m^3 - m^2 + m$.

Let

$$z^m + z^{m-1}\rho_1 + z^{m-2}\rho_2 + \dots + z\rho_{m-1} + \rho_m = 0 \dots\dots (1)$$

be the equation of an algebraical surface referred to rectangular axes x, y, z; ρ_n being an integer function of xy of the degree n

Taking the given point as the origin, the equations to a normal passing through it are

$$\left.\begin{aligned} x + z\frac{dz}{dx} &= 0 \\ y + z\frac{dz}{dy} &= 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots (2).$$

These equations, each of the degree m, combined with equation (1), which is of the same degree, will give rise to a final equation, the degree of which cannot exceed m^3; thus the number of the normals cannot exceed m^3: but this equation involves besides $m^2 - m$ roots which are foreign to the question. To prove this, let us differentiate equation (1) successively with respect to x and z, and with respect to y and z, when we obtain

$$Qdz + Rdx = 0,$$
$$Qdz + Sdy = 0;$$

where Q, R, S are integer functions of x, y, z: from this equations (2) become

$$\left.\begin{aligned} Qx &= Rz \\ Qy &= Sz \end{aligned}\right\} \ldots\ldots\ldots\ldots (3).$$

It is easy to see that if we make $z=0$, the function Q is reduced to ρ_{m-1}, and the first member of equation (1) to ρ_m; therefore, taking the equations

$$z = 0,$$
$$\rho_{m-1} = 0,$$
$$\rho_m = 0,$$

we deduce from them $m(m-1)$ values of x and y, which satisfy the three equations (1) and (3). These values correspond to points of the surface situate in the plane xy, and lines drawn through these points parallel to the axis of z are tangents to the surface. But these points are foreign to the question; therefore the number of normals is $m^3 - m^2 + m$.

Observation 1. The equation of the degree m^3 resulting from the elimination between the three equations (1) and (3), besides the roots foreign to the question, may involve imaginary roots, which in particular positions of the point will still farther reduce the number of possible normals. When the given point is a centre of curvature the equation involves equal roots.

Observation 2. The number of normals to a given surface which can be drawn through a given point, added to the number of tangents which can be drawn to the algebraical curve of the same degree, is always equal to the cube of the degree.

Observation 3. In algebraical curves the final equation is not encumbered with roots foreign to the question, so that the number of normals which can be drawn through a given point to a curve of the degree m, is m^2.

On a Property of Surfaces of the Second Degree: by M. Terquem.

[From Liouville's *Journal de Mathématiques*, vol. IV. p. 241.]

Theorem. In a surface of the second degree the geometrical locus of the points for which the sum of the squares of the normals to the surface is a constant quantity, is a surface of the second degree concentric with the given surface, and having the direction of its principal axes the same.

Let the equation to the given surface referred to its centre be

$$Ax^2 + A'y^2 + A''z^2 + E = 0 \quad\text{.........(1)}.$$

Supposing the axes to be rectangular, let a, b, c be the coordinates of a point, and x', y', z' the coordinates of the point where a normal, passing through the point a, b, c, meets the surface. Then, from the equations to the normal we have the relations

$$\frac{x'-a}{Ax'} = \frac{y'-b}{A'y'} = \frac{z'-c}{A''z'} \quad\text{.. ....(2)}.$$

By eliminating $y'z'$, $x'z'$, $x'y'$, successively between (1) and (2), we obtain

$$x'^6 + 2pax'^5 + x'^4(nc^2 + mb^2 + q) + \text{&c.} = 0,$$
$$y'^6 + 2p'by'^5 + y'^4(n'c^2 + m'a^2 + q') + \text{&c.} = 0,$$
$$z'^6 + 2p''cz'^5 + z'^4(n''b^2 + m''a^2 + q'') + \text{&c.} = 0,$$

where

$$p = \frac{A'}{A - A'} + \frac{A''}{A - A''};\quad p' = \frac{A}{A' - A} + \frac{A''}{A' - A''};$$
$$p'' = \frac{A}{A'' - A} + \frac{A'}{A'' - A'};$$
$$m'' = \frac{AA''}{(A - A'')^2};\quad m = m' = \frac{AA'}{(A - A')^2};$$
$$n = \frac{AA''}{(A - A')^2};\quad n' = n'' = \frac{A'A'}{(A' - A'')^2};$$
$$q = \frac{E}{A};\quad q' = \frac{E}{A'};\quad q'' = \frac{E}{A''}.$$

The sum of the six values of

$$z'^2 = 4p''^2c^2 - 2n''b^2 - 2m''a^2 - 2q'',$$
$$y'^2 = 4p'^2b^2 - 2n'c^2 - 2m'a^2 - 2q',$$
$$x'^2 = 4p^2a^2 - 2nc^2 - 2mb^2 - 2q;$$

and the sum of the six values of

$$-(2ax' + 2by' + 2bz') = 4a^2p + 4b^2p' + 4c^2p''.$$

Now, the square of the normal is

$$(x'-a)^2 + (y'-b)^2 + (z'-c)^2;$$

and representing the sum of the six values of this square by a constant $2R^2$, we obtain after reduction,

$$a^2[2p^2+2p-(m'+m'')+3]+b^2[2p'^2+2p'-(m+n'')+3]$$
$$+c^2[2p''^2+2p'-(n+n')+3]=R^2+q+q'+q'' \ldots\ldots(3).$$

EXAMPLE. If we take the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

equation (3) becomes

$$\left.\begin{array}{l} \frac{x^2}{a^4}[a^4+b^4+c^4-a^2(b^2+c^2)] \\ +\frac{y^2}{b^4}[a^4+b^4+c^4-b^2(a^2+c^2)] \\ +\frac{z^2}{c^4}[a^4+b^4+c^4-c^2(a^2+b^2)] \end{array}\right\}=R^2-(a^2+b^2+c^2);$$

The same method may be followed for surfaces without a centre.

Observation. In lines of the second degree, the geometrical locus of the points for which the sum of the squares of the normal is constant is a similar line, similarly placed.

IV.—ON THE SYMMETRICAL FORM OF THE EQUATION TO THE PARABOLA.

WHEN the parabola is referred to a diameter and the tangent at its vertex, although the equation then assumes the simplest form, yet as these lines are not symmetrical with respect to the curve, the equation itself is not symmetrical with respect to the variables. In order, therefore, to get the equation under a symmetrical form, we must refer the curve to lines similarly situated with respect to it: such are two tangents to the parabola. If we take them as axes, and their intersection as origin, the equation to the curve assumes a form which bears a curious analogy to the symmetrical equations of the other conic sections and of the straight line, and is sufficiently remarkable in itself to deserve attention.

The general equation to a curve of the second degree is

$$(1) \ldots Ay^2+Bxy+Cx^2+Dy+Ex+F=0;$$

the condition that this should represent a parabola is

$$(2) \ldots B^2=4AC \text{ or } B=\pm 2\sqrt{AC},$$

so that (1) is reduced to

$$(3) \ldots (\sqrt{A}y\pm\sqrt{C}x)^2+Dy+Ex+F=0.$$

Now, let the parabola be referred to the two tangents AB, AC (fig. 2.) as axes, and let AB $= a$, AC $= b$, AB being the axis of x, AC of y. Then, since AB is a tangent at B, if we make $y = 0$ in equation (3), the two corresponding values of x must be each equal to a. In this case equation (3) becomes

$$(4) \ldots Cx^2 + Ex + F = 0;$$

and the condition for its being a complete square is

$$(5) \ldots E^2 = 4CF;$$

and as each root is equal to a, we have

$$\frac{F}{C} = a^2, \quad \frac{E}{2C} = -a;$$

$$\text{therefore } C = \frac{F}{a^2}, \quad E = -\frac{2F}{a}.$$

In a similar manner we should find that

$$A = \frac{F}{b^2}, \quad D = -\frac{2F}{b},$$

so that equation (3) takes the form

$$(6) \ldots \left(\frac{y}{b} \pm \frac{x}{a}\right)^2 - 2\left(\frac{y}{b} + \frac{x}{a}\right) + 1 = 0.$$

If we take the superior sign in the first term, this equation is equivalent to

$$\left(\frac{y}{b} + \frac{x}{a} - 1\right)^2 = 0,$$

which is the equation to a straight line, or rather to two which coincide; we must therefore take the inferior sign, in order that the equation may represent a parabola. If, now, we add $\frac{4xy}{ab}$ to both sides, the equation becomes

$$(7) \ldots \left(\frac{y}{b} + \frac{x}{a}\right)^2 - 2\left(\frac{y}{b} + \frac{x}{a}\right) + 1 = \frac{4xy}{ab},$$

the first side of which is a complete square. Extracting, then, the square root on both sides, we have

$$\frac{y}{b} + \frac{x}{a} - 1 = \pm 2\sqrt{\frac{xy}{ab}};$$

or transposing,

$$\frac{y}{b} \mp 2\sqrt{\frac{xy}{ab}} + \frac{x}{a} = 1;$$

the first side of which is also a complete square. Extracting the root again, we finally obtain

$$(8) \ldots \sqrt{\frac{y}{b}} \pm \sqrt{\frac{x}{a}} = \pm 1,$$

which is the required symmetrical form.

The form of this equation shows at once, that the curve lies wholly between the positive axes, as neither x nor y can ever become negative. So long as $x < a$ and $y < b$, the positive signs only on both sides must be taken, as the difference between two fractions can never be unity. If $x > a$ and $y > b$, the negative sign only on the left-hand side must be taken, as the sum of two quantities greater than unity can never be equal to unity; and either sign on the right-hand side, according to the relative magnitude of the terms on the left-hand side. If $y > b$ and $x < a$, the negative sign on the first side and the positive on the second are to be taken; and if $x > a$ and $y < b$, the negative sign on both sides. This apparent discontinuity, which renders it necessary to take sometimes one sign and sometimes another, arises from the equation (8) not being the complete form of the equation to the curve. All the cases are included in the expanded form of (9)

$$\frac{y^2}{b^2} - \frac{2xy}{ab} + \frac{x^2}{a^2} - 2\left(\frac{y}{b} + \frac{x}{a}\right) + 1 = 0.$$

If we transpose one term of (8) and square both sides, we have

$$\frac{y}{b} = 1 \pm 2\sqrt{\frac{x}{a}} + \frac{x}{a},$$

$$\text{or } \frac{y}{b} - \frac{x}{a} = 1 \pm 2\sqrt{\frac{x}{a}},$$

so that $\frac{y}{b} - \frac{x}{a} = 1$ is the equation to a diameter passing through C, and similarly

$$\frac{y}{b} - \frac{x}{a} = -1$$

is the equation to a diameter passing through B, and

$$\frac{y}{b} - \frac{x}{a} = 0$$

to one passing through A.

This form of the equation affords an easy proof of a problem in the Senate-House Papers for 1833. The enunciation is as follows: If there are three tangents to a parabola, the triangle formed by their intersection is half of that whose angular points are the points of contact.

Let ARS, BPC (fig. 2.) be the triangles; then, taking the equation to the parabola referred to AB, AC as axes, the equation to the tangent is

$$\frac{y}{\sqrt{by_1}} + \frac{x}{\sqrt{ax_1}} = 1,$$

where $x_1 y_1$ are the coordinates of the point P.

In this equation, making successively $x = 0$, $y = 0$, we find

$$\text{AS} = \sqrt{by_1}, \quad \text{AR} = \sqrt{ax_1}.$$

Now area ASR $= \frac{1}{2}$ AR . AS sin A $= \frac{1}{2}\sqrt{ab\,x_1y_1}$ sin C,

and area CPB = ACB − NPC − MPB − AMPN.

Now ACB $= \frac{1}{2}ab$ sin C,

NPC $= \frac{1}{2}$ NC . PN sin C $= \frac{1}{2}x_1(b - y_1)$ sin C,

MPB $= \frac{1}{2}$ MB . PM sin C $= \frac{1}{2}y_1(a - x_1)$ sin C,

and AMPN $= x_1y_1$ sin C.

Hence area CPB $= \frac{1}{2}$ sin C $\{ab - x_1(b - y_1) - y_1(a - x_1) - 2x_1y_1\}$

$= \frac{1}{2}$ sin C $(ab - bx_1 - ay_1)$.

But, since x_1y_1 are coordinates of a point in the parabola,

$$\sqrt{\frac{x_1}{a}} + \sqrt{\frac{y_1}{b}} = 1,$$

and therefore

$$\frac{x_1}{a} + 2\sqrt{\frac{x_1y_1}{ab}} + \frac{y_1}{b} = 1;$$

and multiplying by ab, and transposing,

$$2\sqrt{ab\,x_1y_1} = ab - bx_1 - ay_1;$$

so that

$$\text{area CPB} = \tfrac{1}{2}\sin C \,.\, 2\sqrt{ab\,x_1y_1} = \sin C\sqrt{ab\,x_1y_1}$$

and therefore ASR $= \frac{1}{2}$ CPB.

Since AR $= \sqrt{ax_1}$ and AS $= \sqrt{by_1}$, we have, making

AR $= x'$, AS $= y'$,

$$\frac{x'}{a} + \frac{y'}{b} = \frac{\sqrt{ax_1}}{a} + \frac{\sqrt{by_1}}{b} = \sqrt{\frac{x_1}{a}} + \sqrt{\frac{y_1}{b}} = 1 \;\ldots\ldots\; (8);$$

and as the equation to BC is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

x' and y' are coordinates of the line BC; so that if from any point Q in BC we draw QS, QR parallel to the axes, the line joining the points where they cut the axes will be a tangent to the parabola. This gives the means of describing a parabola by the ultimate intersection of a line subject to move under a certain condition. For if

$$\frac{x}{m} + \frac{y}{n} = 1$$

be the equation to RS, m and n are subject to the condition

$$\frac{m}{a} + \frac{n}{b} = 1.$$

G.

V.—ON THE CONDITION OF EQUILIBRIUM OF A SYSTEM OF MUTUALLY ATTRACTIVE FLUID PARTICLES.

THE generally received theory of the Equilibrium of Fluids, (due in its present form to Euler,) assigns one condition as necessary and sufficient in every case. Mr. Ivory conceives, that when a fluid is acted on by forces arising from the mutual attraction of its particles, a second condition is requisite for equilibrium, and has developed the considerations which have led him to this result, in several papers published in the *Phil. Trans.*, and also in the *Phil. Mag.* The authority of Mr. Ivory on any point of mathematical physics is very great: his decision on one to which he has long directed his attention, would be almost final, were it not opposed to the views of Euler, Laplace, and Poisson. The object of this paper is, to examine how far Mr. Ivory, in a paper published in the *Phil. Mag.* vol. XIII., p. 321, has demonstrated the necessity of the subsidiary condition in question. The writer feels it unnecessary to express the diffidence with which he attempts to consider so difficult a subject; he regrets also his inability to discuss Mr. Ivory's views more at large than the present limits would permit.

In the paper just mentioned, Mr. Ivory states the principal steps of the investigation by which Clairaut was led to the condition of equilibrium of a fluid acted on by forces directed to fixed centres; and proceeds to consider the modifications required to adapt the method to the case of a fluid whose particles are mutually attractive. Clairaut first supposes a mass of fluid in equilibrium, and conceives an infinitesimal stratum added to it, which shall produce equable pressure over the whole surface;—the equilibrium of the original mass A will not be disturbed, and the increased mass $A + \delta A$ will be in equilibrio, when the forces acting on its surface are normal to it. This principle, that forces acting on a free surface must be normal to it, was laid down by Huygens, and is confessedly true. By a repetition of this process, the original mass can be enlarged to any extent; and the condition that the nucleus must be in equilibrio becomes, Mr. Ivory observes, unnecessary, by conceiving it diminished *sine limite.* The mathematical condition of equilibrium is, therefore, the expression of the possibility of adding a stratum which shall produce equable pressure, and at the free surface of which the forces shall be normal to it.

Let us endeavour to put this symbolically. Let the force at the original free surface be F; at the point x, y, z produce the normal, and take a length on it $= \frac{\omega}{F}$, ω being infinitesimal: thus we get a stratum producing an equal pressure ω.

Let $f(x, y, z) = c$ be the equation of the free surface; then F being a function of (x, y, z), all that is requisite for the force at any point of the new free surface to be normal is, that

$$f(x, y, z) = c + \delta c$$

shall be its equation.

Let $\qquad V = \sqrt{(f'x)^2 + (f'y)^2 + (f'z)^2}$; then

$$\delta x = \frac{\omega}{F}\frac{f'x}{V}, \quad \delta y = \frac{\omega}{F}\frac{f'y}{V}, \quad \delta z = \frac{\omega}{F}\frac{f'z}{V} \; .... \; (1),$$

and $f(x, y, z) = c = f(x', y', z') - [f'x\,\delta x + f'y\,\delta y + f'z\,\delta z]$

(where $x' = x + \delta x$) by Taylor's theorem;

therefore $c = f(x', y', z') - \frac{\omega}{F}\frac{1}{V}[(f'x)^2 + (f'y)^2 + (f'z)^2]$

$$= f(x', y', z') - \omega\frac{V}{F};$$

$$\text{therefore } c = c + \delta c - \omega\frac{V}{F};$$

or $\frac{V}{F} =$ a constant, which we may take for unity; therefore $F = V$. Resolving this force along the axes,

$X = V\frac{f'x}{V}$; whence $X = f'x$, and so $Y = f'y$, $Z = f'z$.

ω is the increment of pressure $= \delta p$; multiplying the three equations (1) by X, Y, Z, and adding, we get

$$X\delta x + Y\delta y + Z\delta z = \delta p\,\frac{V^2}{F.V};$$

or putting d for δ,

$$dp = X dx + Y dy + Z dz \; ..... \; (2),$$

the equation of equilibrium of an homogeneous and incompressible fluid, whose density is unity.

An objector to Clairaut's reasoning might urge, that this result, though certainly sufficient, was not shown to be necessary: he might argue, that a way has been shown of building up a fluid mass; but that it has not been proved that every fluid mass is capable of resolution into the smaller masses, by means of which alone Clairaut investigates the conditions of equilibrium. Unless it be made a direct postulate, that every fluid mass in equilibrio will continue in equilibrio, when the part of it contained between the free surface and any level surface is removed, it is difficult to see how this objection can be met, except by showing that the property assigned by Huygens to a free surface, viz. that the force is normal to it, belongs to every surface of equal pressure, and that consequently Clairaut's reasoning is in reality independent of any construction or resolution of a fluid mass into successive

strata. When we assert, with Clairaut, that a fluid mass in equilibrium is not disturbed by the addition of a stratum producing equal pressure, we imply that the reaction produced at any point of the surface of A, by the pressures exerted over the rest of the surface, *i. e.* the effect of the transmitted pressures, is normal to it. For we know that the forces at the surface are so; and unless the inference first stated is correct, there could be no equilibrium. It hence appears, that Clairaut's axiom is equivalent to this—Equable pressure produces a reaction normal to the surface on which it is applied. But if the force at a surface of equal pressure were not normal to it, there could be no equilibrium, because it is only by the transmitted pressures that it can be established.

Clairaut, as his views are represented by Mr. Ivory, says nothing of the transmission of pressure; but it is impossible to investigate fluid equilibrium without tacit or expressed reference to some distinctive character of fluidity; and in the principle he makes use of, the idea of the transmission of pressure is essential. It appears, then, that the force at a surface of equal pressure is normal to it; and this conclusion is little else than a different way of putting the principles employed by Clairaut. We are now enabled to dispense with any process of constructing a fluid mass.

On referring to the mathematical reasoning employed above, we shall easily see that, substituting two infinitesimally near surfaces of equal pressure for the consecutive free surfaces of Clairaut, the result we arrive at is simply the symbolical expression of the principle just laid down, viz. that the force at a surface of equal pressure is normal to it. A very little attention will show, that (2) is true in every case of fluid equilibrium, and that it is completely equivalent to the principle which it represents. In translating, so to speak, his fundamental idea from the infinitesimal to the fluxionary conception, that namely of successive generation, Clairaut has tacitly introduced a new condition, namely, that a surface of equal pressure will necessarily be a free surface of equilibrium, the superincumbent part being removed.

Mr. Ivory remarks—"The investigation of Clairaut is clear and definite. It evidently assumes that there is no cause tending to disturb the equilibrium of A, except the action of the forces at the surface of A upon the matter of δA. On this account his method fails when there is a mutual attraction between the mass A and the stratum δA. If the mass A attract the matter of the stratum δA, and cause it to press, it follows necessarily that the matter of δA will react, and by its attraction will urge the particles of A to move from their places. In this case, therefore, the equilibrium of A is disturbed by a force which Clairaut has not attended to; and unless the effect of this new force is counteracted, the body of fluid $A + \delta A$ will not be in equilibrium. The principle of the method suggests a remedy for this omission, for it is easy to prove that the

equilibrium of A will not be disturbed by the attraction of the stratum δA, if the resultant of that attraction on every particle in the surface of A be directed perpendicularly to it."

This reasoning satisfactorily shews, that if a fluid mass of attractive matter be increased by a stratum producing equal pressure over the free surface, the equilibrium will be destroyed unless a certain condition is fulfilled, of which the symbolical expression is

$$c = \int [Pdx + Qdy + Rdz],$$

P, Q, R being the attractions, parallel to the axes of coordinates, of an element of that part of a fluid mass which is external to a given level surface. But the necessity of this condition cannot be proved, unless it is shewn to be impossible in any way to increase the mass A, without destroying the equilibrium, supposing it not fulfilled. All that has been shown is, that the mass cannot be increased by attraction producing equable pressure over the free surface. Now, generally speaking, the mass so increased will not fulfil the condition of having the forces at the new free surface normal to it, those acting at the original free surface being of course so. We cannot, therefore, affirm that we have fallen on a case in which the ordinary condition is fulfilled, without producing equilibrium. If, however, we dispense with the limitation, that the stratum added shall produce equable pressure, we lose the simplicity of Clairaut's method, nor can we make any use of his principle, except by setting aside the construction he employs, which confines him to the particular case in which a surface of equal pressure is potentially a free surface.

This has already been done, and the result is the general equation of equilibrium. It remains to show, that it is in all cases sufficient. It is admitted to be sufficient in the case of a fluid acted on by forces tending to fixed centres. We shall endeavour to reduce the general case to this. Conceive a body acted on by a force directed to a fixed point. It may be so placed, as to remain at rest under the action of the force, that is, the resultant of the force upon it is equal to zero. In this position of the body, the centre of force is some point within it. Let the body, remaining in the same position, diminish *sine limite*, being always similar to itself, the resultant of the force upon it is always equal to zero; and ultimately, when the body becomes a physical point, it coincides in position with the centre of force, and is in the same state with respect to the action of other forces upon it, as if this force did not exist.

This being granted, conceive a homogeneous mass of fluid composed of mutually attractive particles, the free surface of which fulfils the required equation

$$Xdx + Ydy + Zdz = 0.$$

Let the attractive power of each particle be conceived transferred

to a fixed centre of force coinciding with it. Then the action of all the other particles on one particle is precisely replaced by that of the fixed centres; and it has been shown, that the resultant of the action of the centre coinciding with a particle on that particle, equals zero. Hence, the supposition we have made does not change, in any way, the forces acting on any particle of the mass. Were the system in its present and former state respectively to move, the motions would be widely different; but in the arbitrary position we have placed it in, the action on it is precisely the same in the two cases. Now, the single equation given above assures its equilibrium, when we regard it as a system acted on by forces directed to fixed centres; and as the hypothesis by which we are enabled to look upon it in this way nowise affects the forces acting on it, it follows, that the system considered as acted on by mutual attraction must be in equilibrium. Consequently, a mass of homogeneous fluid, the particles of which are mutually attractive, will always be in equilibrium when the free surface fulfils the single condition implied in the general equation obtained above. The same reasoning applies to the case of any fluid, elastic or incompressible.

If this demonstration be thought satisfactory, the question raised by Mr. Ivory, as to the sufficiency of the general equation, must be looked upon as settled. The suggestions here made with respect to the new condition tacitly introduced in Clairaut's reasoning, will, it is thought, enable us to trace the source of the difference of the view taken by Mr. Ivory, and that generally entertained. In one form or other, it seems to recur in every way in which that distinguished mathematician has treated the subject.

R. L. E.

VI.—ON THE CONDITION THAT A SURFACE MAY BE TOUCHED BY A PLANE IN A CURVE LINE.

In Vol. I. p. 83, a demonstration was given of a property of the Wave Surface, that it could be touched by the tangent plane in certain positions in a circle. This is a particular instance of what may be called a singular line in surfaces, analogous to a singular point in curved lines; and when the idea is generalized, it gives rise to the consideration of the possibility of surfaces being touched by a tangent plane in a continuous curve. It is proposed here to investigate the general analytical condition, that any points in a surface should possess this property.

Let $F(x, y, z) = 0$ be the equation to the surface, and put

$$\frac{dF}{dx} = L, \quad \frac{dF}{dy} = M, \quad \frac{dF}{dz} = N.$$

Then the equation to the tangent plane at any point x, y, z, is

$$Lx' + My' + Nz' = Lx + My + Nz.$$

Let the right-hand member of the equation be represented by P; then, if the plane touches the surface in a curve, $\frac{L}{P}$, $\frac{M}{P}$, $\frac{N}{P}$ remain constant while the coordinates vary subject to the condition $F(x, y, z) = 0$, and to another condition, which, together with that, determines the curve. It is this condition which we have to find.

Let $\qquad \frac{d^2F}{dx^2} = l, \qquad \frac{d^2F}{dy^2} = m, \qquad \frac{d^2F}{dz^2} = n,$

$$\frac{d^2F}{dy\,dz} = \lambda, \quad \frac{d^2F}{dx\,dz} = \mu, \quad \frac{d^2F}{dx\,dy} = \nu.$$

Then, as $\frac{L}{P}$, $\frac{M}{P}$, $\frac{N}{P}$ are all constant,

$$\frac{dL}{L} = \frac{dM}{M} = \frac{dN}{N} = \frac{dP}{P} = dQ \text{ suppose.}$$

Or effecting the differentiation indicated,

$$dL = l\,dx + \nu\,dy + \mu\,dz = L\,dQ,$$
$$dM = \nu\,dx + m\,dy + \lambda\,dz = M\,dQ,$$
$$dN = \mu\,dx + \lambda\,dy + n\,dz = N\,dQ.$$

Eliminating dy and dz by cross multiplication,

$$R\,dx = \{L(mn - \lambda^2) + M(\lambda\mu - n\nu) + N(\lambda\nu - m\mu)\}\,dQ;$$

where $\quad R = lmn - (l\lambda^2 + m\mu^2 + n\nu^2) + 2\lambda\mu\nu,$

a symmetrical function of $l, m, n, \lambda, \mu, \nu$. Similarly we have

$$R\,dy = \{L(\lambda\mu - n\nu) + M(nl - \mu^2) + N(\mu\nu - l\lambda)\}\,dQ,$$
$$R\,dz = \{L(\lambda\nu - m\mu) + M(\mu\nu - l\lambda) + N(lm - \nu^2)\}\,dQ.$$

But from the equation to the surface we have also the condition

$$L\,dx + M\,dy + N\,dz = 0.$$

Therefore, multiplying the previous equations by L, M, N respectively, and adding, the first side of the equation disappears by the last condition, and we have

$$L^2(mn-\lambda^2) + M^2(ln-\mu^2) + N^2(lm-\nu^2) + 2MN(\mu\nu-l\lambda)$$
$$+ 2LN(\lambda\nu-m\mu) + 2LM(\lambda\mu-n\nu) = 0;$$

which equation, combined with $F(x, y, z) = 0$, determines the

curve of contact: and this condition must subsist in order that the surface may be touched by the tangent plane in a curve.

This expression may be reduced into a simpler shape, by supposing the original equation put into the form

$$f(x, y) - z = 0,$$

and employing the partial differential coefficients of z. We have then

$$\mathrm{L} = p, \quad \mathrm{M} = q, \quad \mathrm{N} = -1, \quad l = r, \quad m = t, \quad n = 0,$$
$$\lambda = 0, \quad \mu = 0, \quad \nu = s\,;$$

and substituting these values in the equation, it is reduced to

$$rt - s^2 = 0.$$

This is the condition which subsists for every point of developable surfaces, as is easily seen ought to be the case, since in their case the tangent plane at every point touches them along a straight line.

S. S. G.

VII.—ON SINGULAR SOLUTIONS AND PARTICULAR INTEGRALS OF DIFFERENTIAL EQUATIONS.*

1. Any differential equation, of the n^{th} order and of the r^{th} degree in respect to the highest of its differential coefficients, may be conceived as resolved into r factors, each of the n^{th} order and of the first degree with regard to this differential coefficient, and the satisfaction of the compound equation will depend upon the satisfaction of the separate equations arising from putting each of these factors equal to zero. It will therefore be sufficient for us, in the following investigations, to discuss the nature of the singular solutions and the particular integrals of equations involving the highest differential coefficient of the first degree only.

Any differential equation, although it be multiplied by any function of the variables not involving the highest of the differential coefficients, is still regarded as the same differential equation.

2. Let $f\{x, y, y', \ldots y^{(n)}\} = 0$, represent any differential equation of the n^{th} order and of the first degree in $y^{(n)}$. The following are evidently the only functions of $x, y, y', \ldots y^{(n-1)}$ which can satisfy this identity.

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* From a Correspondent.

(α) A function, of which the differential coefficient is a factor of the differential equation, and of which the magnitude is any quantity, the variation of which is independent of the variation of the variables involved in the function. Let V represent such a function—then the identical equation $V = c$ is called the complete primitive, or a first integral of the differential equation, accordingly as the differential equation is of the first or of a higher order; and c, which represents the arbitrary magnitude of the function, is called the arbitrary constant. We shall always suppose that V contains no term independent of the variables, since any such term might be comprehended in the arbitrary constant. We will also use the term 'regular integral' to comprehend both complete primitives and first integrals. Every differential equation of the n^{th} order has (n) regular integrals.

(β) A function v of an assigned value (a), such that, after the differential equation has been so prepared that $v - a$ is not a factor of the whole of it, the differential coefficient of v is a factor of one portion of it, and $v - a$ of another.

The identity $v = a$ is called a particular integral, or a singular solution of the differential equation, accordingly as the relation which it establishes among the variables be or be not a result of the imposition of some definite value upon the arbitrary constant of the regular integral.

The term particular integral is applied also to identical equations belonging to functions of definite magnitude, which satisfy the differential equation precisely in the manner of the regular integral.

(γ) A function of invariable and definite magnitude which, by the relation which it establishes among the variables by the definiteness of its value, renders identically equal to zero a factor of the whole differential equation. If this function be not coincident with any of the functions of the second case, it is regarded as a factor foreign to the equation, and the relation which it establishes among the variables is not reckoned a solution.

3. Let $Z = 0$ represent a singular solution of a differential equation of the n^{th} order and of the first degree. The function Z will, as we know by the theory of equations, be equivalent to the product of a number of functions, in each of which the differential coefficient of highest order appears only in the first power, and the equating to zero of each of these factors will give us all the values of the highest differential coefficient in terms of the other variables. Hence, clearly, if v be any one of these factors, $v = 0$ will be a singular solution.

Let $c = \eta$ be a regular integral. Then, since the relation $v = 0$ is incompatible with a constant value for c, if we conjoin the relation $v = 0$ with the equation $c = \eta$, we shall get c equal to some function of $x, y, y', \ldots y^{(n-1)}$. Hence, evidently, if $v = 0$ be a singular solution, the most general expression for η may be re-

presented by $wv^{\alpha} + u$, where w and u are some functions of $x, y, y', \ldots y^{(n-1)}$; and w has been so chosen that no term in u contains v either as a factor or as a divisor, and where the a, having been so chosen that w becomes neither zero nor infinity for the relation $v = 0$, is a positive quantity.

Differentiating the regular integral, we get

$$0 = a\,v^{\alpha-1}\,w\,dv + v^{\alpha}\,dw + du,$$

$$\text{and } \therefore\ 0 = dv + \frac{v^{1-\alpha}}{a}\left(v^{\alpha}\frac{dw}{w} + \frac{du}{w}\right);$$

but, since $v = 0$ is a singular solution of this equation, it is plain that a must be less than unity, since otherwise the equation would not be satisfied. Hence, if $c = wv^{\alpha} + u$ be a regular integral of a differential equation, the a being a positive quantity less than unity, $v = 0$ will be a singular solution of the differential equation.

Ex. If $c = (x + y + 1)^{\frac{1}{2}} + x$ be the complete integral of a differential equation, $x + y = -1$ or $x + y + 1 = 0$ will be a singular solution. Thus, differentiating, we have

$$0 = \frac{dx + dy}{2(x + y + 1)^{\frac{1}{2}}} + dx \text{ .... } (l),$$

$$\text{and } \therefore\ 0 = dx + dy + 2(x + y + 1)^{\frac{1}{2}}\,dx \ldots (k),$$

which is satisfied by $x + y = -1$, a relation incompatible with any constant value for c in the integral.

4. The factor which renders the differential equation

$$0 = dv + \frac{v^{1-a}}{a}\left(v^{\alpha}\frac{dw}{w} + \frac{du}{w}\right)$$

a perfect differential, is $awv^{\alpha-1}$ or $\dfrac{aw}{v^{1-a}}$, which $= \infty$ when $v = 0$. Hence we see that the integrating factor of a differential equation becomes equal to infinity for the relation between $x, y, y', \ldots$, expressed by a singular solution.

Thus, the integrating factor for equation (k) in the preceding article is $\dfrac{1}{2(x+y+1)^{\frac{1}{2}}}$, which $= \infty$ for the relation $x+y+1=0$, which constitutes the singular solution.

5. It is clear that $v = 0$ does not satisfy the equation

$$0 = a\,v^{\alpha-1}\,w\,dv + v^{\alpha}\,dw + du$$

Hence we see, that every differential equation may be so prepared as to become insusceptible of any assigned singular solution; and likewise, that a state of perfect differentiality is an instance of this. Thus the equation (l) in Art. (3) is not satisfied by the singular solution $x + y + 1 = 0$.

6. We may write the differential equation in the form

$$dv + \frac{v}{a}\frac{dw}{w} + \frac{v^{1-a}}{a}\frac{du}{w} = 0.$$

Here we see that $v = 0$ satisfies the equation when a is a negative quantity, as well as when it is a positive quantity less than (a); but $c = wv^a + u$ becomes $= \infty$ in this case, which shews, that whenever such an equation is satisfied by $v = 0$, this must be a particular integral derivable from the regular integral, by putting the arbitrary constant equal to infinity.

Ex. Let $\qquad c = y(x^2 + y + a)^{-1} + by$

be the complete primitive.

The differential equation may be written

$$0 = (x^2 + y + a)\,\{dy + b\,dy\,(x^2 + y + a)\} - y\,(2x\,dx + dy),$$

and is satisfied by $x^2 + y + a = 0$, which gives $c = \infty$.

7. It may be that several singular solutions, such as $v = 0$, may correspond to the single value u of c.

Let $v_1 = 0$, $v_2 = 0$, ... $v_n = 0$, denote all the singular solutions of this class. Then clearly

$$c = \mu \,.\, \phi(u) \,.\, v_1^{a_1} \,.\, v_2^{a_2} \dots v_n^{a_n} + u,$$

where $a_1, a_2, \dots a_n$ are all positive quantities less than unity, where $\phi(u)$ is some function of u, and where μ is some function of the variables, such that $\mu = 0$ does not constitute a singular solution. Hence the most comprehensive expression for a singular solution for the value u of c in this case is

$$v_1^{\beta_1} \,.\, v^{\beta_2} \dots v_n^{\beta_n} = 0,$$

where $\beta_1, \beta_2, \dots \beta_n$, are any positive quantities.

Ex. For instance, let the complete integral of a differential equation be

$$c = (xy^2 + 1)^2 (x + y)^{\frac{1}{3}} (x^2 + y^2 - a^2)^{\frac{1}{2}} (x - y)^{\frac{1}{4}} (x^3 - y)^{-1} + 3xy^2 + 1.$$

Then we know at once that the only singular solutions connected with the differential equation to which it belongs are

$$x + y = 0,$$
$$x^2 + y^2 - a^2 = 0,$$
$$x - y = 0;$$

or any product of these, for instance,

$$(x + y)^2 (x^2 + y^2 - a^2) = 0.$$

Again, $x^3 - y = 0$ is a particular integral corresponding to $c = \infty$. Also, $xy^2 + 1 = 0$ is a particular integral for $c = -2$.

8. Let $c = wv^a + u$ be the regular integral of a differential equation, where the same character belongs to the symbols in-

volved as in Art. 3, except that here a is $=$ or $>$ 1, and a positive quantity. Differentiating we get

$$0 = awv^{a-1}dv + v^a dw + du.$$

Now, if we suppose $v = 0$, we get $du = 0$; but from the arrangements which were made about u this is impossible. Hence, when a is $=$ or > 1, $v = 0$ is no solution at all.

Ex. In equation $c = (x+y)^{\frac{3}{2}} . (x-y)^{\frac{1}{2}} + x$,

$x + y = 0$ is no solution at all,

$x - y = 0$ is a singular solution.

9. Differentiating the equation $c = wv^a + u$ with respect to x alone, we have

$$\frac{dc}{dx} = awv^{a-1}\frac{dv}{dx} + v^a\frac{dw}{dx} + \frac{du}{dx}.$$

Now, $\frac{dv}{dx}$ is not generally $= 0$ for all the simultaneous values of the variables of the equation $v = 0$. Hence, if $v = 0$ be either a singular solution or a particular integral, deducible from the regular integral by putting $c = \infty$, the imposition of the relation expressed by $v = 0$ upon the variables in the expression for $\frac{dc}{dx}$ will render it equal to infinity. In just the same way we may shew that $\frac{dc}{dy}$, $\frac{dc}{dy'}$, ..., for the relation $v = 0$, become all of them equal to infinity.

10. We will now proceed to shew, that every first integral of a differential equation gives rise to the same singular solutions.

Let $v = 0$ be a singular solution belonging to a first integral $c = wv^a + u$ of a differential equation

$$dv + \frac{v^{1-a}}{a}\left(v^a\frac{dw}{w} + \frac{du}{w}\right) = 0.$$

If this be not a singular solution connected with any other of the first integrals, it must be a particular integral, since it does satisfy the differential equation. Hence this other first integral must evidently be expressible under the form

$$c' = \mu v^{\beta} + b,$$

where b is some definite quantity, and where we will suppose β has been so chosen that μ does not equal 0 or ∞ for the relation $v = 0$. Differentiating this equation, we get

$$dv + \frac{v}{\beta}\frac{d\mu}{\mu} = 0;$$

and, since the differential equation must be the same for both first integrals, it follows that

$$\frac{v}{\beta}\frac{d\mu}{\mu} = \frac{v^{1-a}}{a}\left(v^a \frac{dw}{w} + \frac{du}{w}\right);$$

therefore $$\frac{v}{\beta}\frac{d\mu}{\mu} = \frac{v}{a}\frac{dw}{w} + \frac{v^{1-a}}{a}\frac{du}{w};$$

therefore $$v^a\left(\frac{1}{\beta}\frac{d\mu}{\mu} - \frac{1}{a}\frac{dw}{w}\right) = \frac{1}{a}\frac{du}{w};$$

therefore $$v^a = \frac{\frac{1}{a}\frac{du}{w}}{\frac{1}{\beta}\frac{d\mu}{\mu} - \frac{1}{a}\frac{dw}{w}}.$$

Hence, when $v = 0$, its equivalent $\frac{1}{a}\frac{du}{w}\left(\frac{1}{\beta}\frac{d\mu}{\mu} - \frac{1}{a}\frac{dw}{w}\right)^{-1}$ must also become equal to 0; but, from the arrangements which have been made with regard to the elements of this expression, it is clear that this condition cannot be satisfied for all the simultaneous values of the variables expressed by the equation $v = 0$. Hence we see that $v = 0$ must likewise be a singular solution with regard to every other first integral, or, in other words, that every first integral must give rise to the same singular solutions.

Ex. Let the complete integral of a differential equation of the second order be

$$c = (1 + a^2)x + ay^2 + x^2.$$

From which $0 = 1 + a^2 + 2ayy' + 2x$;

therefore $a^2 + 2ayy' + y^2y'^2 = y^2y'^2 - 2x - 1$;

whence $a = -yy' + (y^2y'^2 - 2x - 1)^{\frac{1}{2}}$;

therefore $y^2y'^2 - 2x - 1 = 0$ is a singular solution for one first integral, but

$$\begin{aligned} c &= (1 + a^2)x + ay^2 + x^2, \\ &= -x(2ayy' + 2x) + ay^2 + x^2, \\ &= a(y^2 - 2xyy') - x^2, \\ &= \{(y^2y'^2 - 2x - 1)^{\frac{1}{2}} - yy'\}(y^2 - 2xyy') - x^2; \end{aligned}$$

from which it is evident that $y^2y'^2 - 2x - 1 = 0$ is a singular solution for the other first integral.

11. Let $y = k$ be one of the values of y corresponding to a singular solution of a differential equation of the first order, k being some function of x.

Let the complete primitive be expressed under the form

$$c = w(y - k)^a + u,$$

where a and w have been so chosen, that u contains no term involving $y - k$ as a factor, and that w is of the form

$$\rho_0 + \rho_1(y - k)^{\beta_1} + \rho_2(y - k)^{\beta_2} + \ldots$$

where $\rho_0, \rho_1, \rho_2, \ldots$ have not any of them $y-k$ as a constituent factor, and where, since $y=k$ belongs to a singular solution, $\beta_1, \beta_2, \ldots$ are all positive quantities.

Differentiating with respect to x, we get

$$0 = \frac{dw}{dx}(y-k)^a - aw(y-k)^{a-1}\frac{dk}{dx} + \frac{du}{dx}$$
$$+ p\left\{\frac{dw}{dy}(y-k)^a + aw(y-k)^{a-1} + \frac{du}{dy}\right\},$$

and therefore

$$-p = \frac{\dfrac{dw}{dx}(y-k)^a - aw(y-k)^{a-1}\dfrac{dk}{dx} + \dfrac{du}{dx}}{\dfrac{dw}{dy}(y-k)^a + aw(y-k)^{a-1} + \dfrac{du}{dy}},$$

$$= \frac{\dfrac{dw}{dx}(y-k) - aw\dfrac{dk}{dx} + \dfrac{du}{dx}(y-k)^{1-a}}{\dfrac{dw}{dy}(y-k) + aw + \dfrac{du}{dy}(y-k)^{1-a}},$$

therefore

$$-p+\frac{dk}{dx} = \frac{\left(\dfrac{dw}{dx} + \dfrac{dw}{dy}\dfrac{dk}{dx}\right)(y-k) + \left(\dfrac{du}{dx} + \dfrac{du}{dy}\dfrac{dk}{dx}\right)(y-k)^{1-a}}{aw + \dfrac{dw}{dy}(y-k) + \dfrac{du}{dy}(y-k)^{1-a}},$$

but $\dfrac{dw}{dx} + \dfrac{dw}{dy}\dfrac{dk}{dx} = \dfrac{d\rho_0}{dx} + \dfrac{d\rho_0}{dy}\dfrac{dk}{dx} + \left(\dfrac{d\rho_1}{dx} + \dfrac{d\rho_1}{dy}\dfrac{dk}{dx}\right)(y-k)^{\beta_1} + \ldots$

Hence it is plain that we may write

$$p - \frac{dk}{dx} = \frac{P(y-k)^{1-a} + Q(y-k)}{R + S(y-k)^{\gamma}},$$

where P, Q, R do not involve $y-k$ as a factor. Consequently, in the development of p for a substitution of $k+h$ for y, where h is an arbitrary quantity, and $y=k$ belongs to a singular solution, the index of the lowest power of h is fractional, and therefore $\dfrac{dp}{dy} = \infty$ for a singular solution

If $y-k=0$ had belonged to a constant value of c, not infinity, we should have got

$$-p + \frac{dk}{dx} = \frac{\left(\dfrac{dw}{dx} + \dfrac{dw}{dy}\dfrac{dk}{dx}\right)(y-k)}{aw + \dfrac{dw}{dy}(y-k)},$$

and the expansion might have been effected in powers of h, of which the lowest index would have been not less than unity. Hence $\dfrac{dp}{dy}$ would not $=\infty$.

Again, if $y - k = 0$ had corresponded to an infinite value for c, we should have got

$$-p + \frac{dk}{dx} = \frac{\left(\frac{dw}{dx} + \frac{dw}{dy}\frac{dk}{dx}\right)(y-k) + \left(\frac{du}{dx} + \frac{du}{dy}\frac{dk}{dx}\right)(y-k)^{1+\alpha}}{aw + \frac{dw}{dy}(y-k) + \frac{du}{dy}(y-k)^{1+\alpha}},$$

and in the development by powers of h the lowest index of h would have been unity. Hence we see that $\frac{dp}{dy}$ would not $= \infty$.

From these results we see, that whenever we discover a solution of a differential equation, we may ascertain whether or not it is a singular solution by trying whether it will render $\frac{dp}{dy} = \infty$.

It must be remarked, that if we determine $\frac{dp}{dy}$ from a differential equation, and make the result $= \infty$, it does not follow that the relation between x and y, resulting from this condition, will be a singular solution—it may not be a solution at all; but if it be a solution, as we can ascertain by seeing whether it satisfies the equation, it must be a singular solution. So that if there be any singular solution or singular solutions, we are certain to detect them by this method.

Ex. 1. Let us take the differential equation

$$0 = (x^2 - y)\,dy + 3x\,dx,$$

$$\text{we get } \frac{dy}{dx} = \frac{3x}{y - x^2};$$

$$\text{therefore } \frac{d}{dy}\frac{dy}{dx} = -\frac{3x}{(y - x^2)^2} = \infty \text{ when } y = x^2;$$

but $y = x^2$ does not satisfy the differential equation, and is no solution at all.

Ex. 2. Let us take the differential equation

$$\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} + x = 0,$$

$y^2 + (x - 1)^2 = 0$ satisfies this equation.

Solving it in $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = -\tfrac{1}{2}y + \tfrac{1}{2}\sqrt{y^2 - 4x};$$

$$\text{therefore } \frac{d}{dy}\frac{dy}{dx} = -\tfrac{1}{2} + \tfrac{1}{2}\frac{y}{\sqrt{y^2 - 4x}},$$

which does not $= \infty$, for $y^2 + (x - 1)^2 = 0$.

Hence, $y^2 + (x - 1)^2 = 0$ is a particular integral.

Again, equating $y^2 - 4x$ to zero, we get $y = \pm 2\sqrt{x}$; and substituting this in our differential equation, we have

$$x + 2x + x = 0, \text{ an absurdity.}$$

Hence the differential equation has no singular solutions.

Ex. 3. If from the equation

$$x\frac{dy}{dx} - y = y\frac{dy}{dx} + (a - x)\frac{dy^2}{dx^2},$$

we determine $\frac{d}{dy}\frac{dy}{dx}$, and equate it to infinity, we shall get

$$(x + y)^2 - 4ay = 0,$$

and this satisfies the equation. Hence it is a singular solution.

12. Let

$$c = \phi(u) \cdot \mu \cdot v_1^{a_1} \cdot v_2^{a_2} \ldots\ldots v_n^{a_n} + u$$

be a regular integral of a differential equation, the symbols here involved being such as in Art. 7.

From this equation suppose that we obtained another shape of the regular integral $V = f(x, y, y', \ldots y^{(n-1)}, c) = 0$; and let us suppose this function to be such, that $V_{c=u} = 0$ comprehends no relations among the variables, except those of the singular solutions $v_1 = 0$, $v_2 = 0$, &c. Then clearly V must be such a function of the variables, that when u is substituted in it for c, it shall be reduced to $v_1^{\beta_1} . v_2^{\beta_2} \ldots v_n^{\beta_n}$, where $\beta_1, \beta_2, \ldots \beta_n$ are all positive quantities.

Hence, clearly,

$$V = (v_1^{\beta_1} . v_2^{\beta_2} \ldots v_n^{\beta_n})^\lambda - \left\{\frac{(c - u)\mu}{\phi(u)} . v_1^{\beta_1 - a_1} . v_2^{\beta_2 - a_2} \ldots v_n^{\beta_n - a_n}\right\}^\lambda,$$

where λ is some positive quantity.

From this we get

$$\frac{dV}{dc} = -\lambda(c - u)^{\lambda - 1} . \left(\frac{\mu}{\phi(u)} . v_1^{\beta_1 - a_1} \ldots v_n^{\beta_n - a_n}\right)^\lambda,$$

$$= 0 \text{ when } c = u, \text{ provided that } \lambda > 1.$$

Hence we see, that if $V = 0$ be a regular integral of a differential equation, involving the arbitrary constant c to a higher power than the first, we may sometimes get a singular solution by obtaining a value of c from the equation $\frac{dV}{dc} = 0$, and substituting it in the equation $V = 0$.

If V had been

$$= [\mu^\gamma \{\phi(u)\}^\delta . v_1^{\beta_1} . v_2^{\beta_2} \ldots v_n^{\beta_n}]^\lambda -$$
$$- [(c - u)\mu^{1+\gamma} . \{\phi(u)\}^{1+\delta} . v_1^{\beta_1 - a_1} \ldots v_n^{\beta_n - a_n}],$$

then clearly $\frac{dV}{dc} = 0$ would have given us, not only singular solutions, but also solutions $\mu = 0$, $\phi(u) = 0$, of which the former are no solutions at all, and the latter particular integrals.

Ex. Let $c = (x + y)(x - y)^{\frac{1}{2}} + x$,

$$(c - x)^2 = (x + y)^2 (x - y),$$

$$V = (c - x)^2 - (x + y)^2 (x - y).$$

Put $$\frac{dV}{dc} = 2(c - x) = 0;$$

$$\text{therefore } c = x,$$

and we have $$(x + y)^2 (x - y) = 0.$$

Now the differential equation is

$$0 = (dx + dy)(x - y)^{\frac{1}{2}} + (x + y)\frac{dx - dy}{2(x - y)^{\frac{1}{2}}} + dx,$$

and $x + y = 0$ does not satisfy the equation.

Hence $(x+y)^2 (x-y)=0$ gives not only a relation $x-y=0$, constituting a singular solution, but also $x + y = 0$, which is no solution at all. Thus we see, that unless we can obtain the differential equation, or, which is the same thing, determine another form of $V = 0$, where the arbitrary constant is explicit, we do not know whether $\frac{dV}{dc} = 0$ will lead to any kind of solution or not. And if we *can* get c explicit, we can always see at once what the singular solutions are.

13. Let $f(x, y, c) = 0$ be the complete primitive of a differential equation of the first order, the arbitrary constant being involved among the variables.

Differentiating, we get

$$\frac{df(x, y, c)}{dx} + \frac{df(x, y, c)}{dy}\frac{dy}{dx} = 0;$$

but since c is a function of the two independent quantities x and y, we have

$$\frac{df(x, y, c)}{dx} + \frac{df(x, y, c)}{dc}\frac{dc}{dx} = 0,$$

and $$\frac{df(x, y, c)}{dy} + \frac{df(x, y, c)}{dc}\frac{dc}{dy} = 0;$$

and from these three equations we get

$$\frac{df(x, y, c)}{dx} + \frac{df(x, y, c)}{dy}\frac{dy}{dx} = -\frac{dV}{dc}\left(\frac{dc}{dx} + \frac{dc}{dy}\frac{dy}{dx}\right) \ldots(k).$$

Suppose next that c has some variable value u, incompatible

with the expression for c, resulting from the solution of $f(x,y,c)=0$ in respect to c. Differentiating equation $f(x, y, u) = 0$, we have

$$\frac{df(x, y, u)}{dx} + \frac{df(x, y, u)}{dy}\frac{dy}{dx} + \frac{df(x, y, u)}{du}\left(\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}\right) = 0.$$

Suppose, now, that $\frac{df(x, y, u)}{du} = 0$; then, since $\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx}$ cannot be generally equal to an infinite quantity, since u does not involve as a factor the function, the vanishing of which reduced c to u, we shall have

$$\frac{df(x, y, u)}{dx} + \frac{df(x, y, u)}{dy}\frac{dy}{dx} = 0;$$

but u is involved in $f(x, y, u) = 0$, just in the same way as c was in $f(x, y, c) = 0$; and therefore from these two equations we shall get the same expressions for u and for c. Consequently the equation resulting from the differentiation of $f(x, y, c) = 0$, and the subsequent elimination of c is satisfied by the equation $f(x, y, u) = 0$, provided that $\frac{df(x, y, u)}{du} = 0$. We must not, however, be led to infer from this, that $f(x, y, u)=0$ is necessarily a solution of the differential equation belonging to the complete primitive $f(x, y, c) = 0$; for by equation (k) we see, that the differential equation which is satisfied by $f(x, y, u) = 0$ is the differential equation in its state of perfect differentiality multiplied by $\frac{df(x, y, u)}{du}$, and therefore its satisfaction may result from the fact, that $\frac{df(x, y, u)}{du}$ has been made $= 0$ quite independently of its own peculiar constitution.

The same remarks are applicable to a first integral

$$f(x, y, y', \dots y^{(n-1)}c) = 0$$

of any equation.

Ex. Let $c = (x + y)^2 + y$ be the complete primitive of a differential equation.

$$(c - y)^2 - (x + y)^4 = 0 = f(x, y, c) \text{ suppose,}$$

$$\frac{df(x, y, c)}{dc} = 2(c - y) = 0 \text{ suppose;}$$

$$\text{therefore } c = y,$$

$$\text{therefore } f(x, y, u) = -(x + y)^4;$$

$$\text{but } \frac{df(x, y, c)}{dx} + \frac{df(x, y, c)}{dy}\frac{dy}{dx} = 0,$$

$$\text{therefore } -4(x + y)^3 - \{2(c - y) + 4(x + y)\}\frac{dy}{dx} = 0,$$

$$\text{and} \quad -4(x+y)^3 - \{2(x+y)^2 + 4(x+y)\}\frac{dy}{dx} = 0,$$

and this is *satisfied* by $x + y = 0$,

but $x + y$ is a factor. Dividing out by it, we get

$$-4(x+y)^2 - \{2(x+y) + 4\}\frac{dy}{dx} = 0,$$

which is not satisfied by $x + y = 0$, and therefore $(x + y)^4 = 0$ or $x + y = 0$ is not a solution.

W. W.

VIII.—ON SOME EXPRESSIONS FOR THE AREA OF A TRIANGLE.

The expressions for the area of a triangle are usually given in terms of the sides and angles which, being the fundamental parts of the figure, are naturally the quantities of which every expression relating to the triangle is made a function. It is, however, possible to express the area of a triangle in terms of other independents (if we may use the phrase); and as in two cases the results are remarkable for simplicity, we shall here proceed to investigate them. The independent quantities which we shall assume in the first case, are the lines joining the angles with the middle points of the opposite sides, and it will be seen that the form of the expression is exactly the same as that involving the sides.

1. Let ABC (fig. 3.) be a triangle, D, E, F the middle points of the sides; join AD, BE, CF, which will all pass through one point O, such that $\text{OD} = \frac{1}{2}\text{OA}$, $\text{OE} = \frac{1}{2}\text{OB}$, $\text{OF} = \frac{1}{2}\text{OC}$. Produce BO to G, and make $\text{OG} = \text{BO}$, so that $\text{OE} = \text{EG}$, and join AG, CG. Now, since $\text{AE} = \text{CE}$ and $\text{OE} = \text{EG}$, the triangles AEO, CEG are equal in every respect, and hence CG is parallel and equal to AO; similarly, AG is parallel and equal to OC. Now, the three triangles BOC, AOC, AOB, being all equal, each is equal to $\frac{1}{3}\text{ABC}$; but $\text{AOG} = \text{AOC}$, as they are on the same base and between the same parallels; therefore $\text{AOG} = \frac{1}{3}\text{ABC}$.

Let $\text{AO} = \alpha$, $\text{BO} = \text{OG} = \beta$, $\text{OC} = \text{AG} = \gamma$. Then

$$\text{area AOG} = \sqrt{\frac{(\alpha+\beta+\gamma)}{2}\,\frac{(\alpha+\beta-\gamma)}{2}\,\frac{(\alpha+\gamma-\beta)}{2}\,\frac{(\beta+\gamma-\alpha)}{2}}.$$

Let $\text{AD} = h$, $\text{BE} = k$, $\text{CF} = l$. Then

$$\alpha = \tfrac{2}{3}h, \quad \beta = \tfrac{2}{3}k, \quad \gamma = \tfrac{2}{3}l:$$

substituting these values in the previous expression,

$$\text{area AOG} = \sqrt{\frac{(h+k+l)}{3}\,\frac{(h+k-l)}{3}\,\frac{(h+l-k)}{3}\,\frac{(l+k-h)}{3}};$$

and as area ABC = 3 area AOG,

$$\text{area ABC} = \tfrac{1}{3}\sqrt{(h+k+l)\,(h+k-l)\,(h+l-k)\,(k+l-h)}.$$

If we put $h+k+l=2s$, and transform accordingly the other factors, we find

$$\text{area ABC} = \tfrac{4}{3}\sqrt{s\,(s-h)\,(s-k)\,(s-l)},$$

which is of the same form as the expression for the area in terms of the sides.

2. We cannot obtain a similar expression in terms of the perpendiculars from the angles on the opposite sides; but if we avail ourselves of the relations between these and the radii of the circles touching the sides, given in vol. I. p. 21, we obtain a very simple expression for the area of the triangle.

Let p, q, r be the three perpendiculars on the sides a, b, c respectively; then, if A be the area of the triangle,

$$a = \frac{2\text{A}}{p}, \quad b = \frac{2\text{A}}{q}, \quad c = \frac{2\text{A}}{r}.$$

$$\text{Now, A} = \sqrt{\frac{(a+b+c)}{2}\,\frac{(a+b-c)}{2}\,\frac{(a+c-b)}{2}\,\frac{(b+c-a)}{2}};$$

and substituting for a, b, c their values,

$$\text{A}=\text{A}^2\sqrt{\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)\left(\frac{1}{p}+\frac{1}{r}-\frac{1}{q}\right)\left(\frac{1}{q}+\frac{1}{r}-\frac{1}{p}\right)}.$$

But if ρ be the radius of the inscribed circle, ρ_1, ρ_2, ρ_3 the radii of the circles which touch two of the sides internally and one externally, it was shown, in the article referred to above, that

$$\frac{1}{\rho}=\frac{1}{p}+\frac{1}{q}+\frac{1}{r}, \quad \frac{1}{\rho_1}=\frac{1}{p}+\frac{1}{q}-\frac{1}{r},$$

$$\frac{1}{\rho_2}=\frac{1}{p}+\frac{1}{r}-\frac{1}{q}, \quad \frac{1}{\rho_3}=\frac{1}{q}+\frac{1}{r}-\frac{1}{p};$$

$$\text{therefore A} = \frac{\text{A}^2}{\sqrt{\rho\rho_1\rho_2\rho_3}},$$

$$\text{or A} = \sqrt{\rho\rho_1\rho_2\rho_3}.$$

From this we can obtain easily an expression for the perimeter of the triangle in terms of these radii. For since we have $\rho(a+b+c)=2\text{A}$, it follows that

$$a+b+c = 2\sqrt{\frac{\rho_1\rho_2\rho_3}{\rho}}.$$

W.

IX.—ON THE METHOD OF SPHERICAL COORDINATES. No. II.

We shall now proceed, in continuation of Art. I. of No. V., to further exemplifications of the method of spherical coordinates, applying it first to find the areas and lengths of spherical curves.

16. Let $C\phi$, $C\psi$ (fig. 4.) be the coordinate axes, and let two consecutive ordinates and parallels to $C\phi$ form a small rectangular area, which may ultimately be considered as a plane area bounded by rectilinear sides. Then

The sides of this parallel to $C\phi$ will be $\cos\psi\, d\phi$,
.... $C\psi$ .... $d\psi$,

and therefore the element of the area will be $\cos\psi\, d\psi\, d\phi$; so that if A be the area,

$$A = \iint \cos\psi\, d\psi\, d\phi = \int \sin\psi\, d\phi \text{ .... } (1).$$

If, therefore, we have, by means of the equation to the curve, a relation between ψ and ϕ, by substituting for ψ its value in terms of ϕ, or conversely, and integrating between the proper limits, we can determine the area.

As an example, let us take the curve whose equation is

$$\psi = \phi.$$

$$\text{Then} \quad A = \int \sin\phi\, d\phi = C - \cos\phi;$$

which, taken from $\phi = 0$ to $\phi = \frac{\pi}{2}$, gives $A = 1$, A being here the fourth part of the area included between the curve and the axis of ϕ. The area included between the curve and the axis of ψ is $\frac{\pi}{2} - 1$, and therefore the whole area enclosed by the curve being four times this, is $2\pi - 4$; which, subtracted from the hemisphere, gives for the residue the square of the diameter of the sphere. Now, the curve whose equation is $\phi = \psi$, is by (12) that produced by the intersection with the sphere of a cylinder whose radius is half the radius of the sphere, and whose circumference passes through the centre of the sphere; therefore this cylinder cuts off from the sphere such an area, that the residue of the hemisphere is quadrable. This celebrated proposition was proposed by Viviani as a challenge to the mathematicians of his day, and was solved by James Bernouilli.

17. If we refer the curves to polar coordinates ϕ and θ, we have for the sides of the elemental area

$$d\phi \text{ and } \sin\phi\, d\theta;$$

and therefore if A be the area,

$$\begin{aligned} A &= \iint \sin\phi\, d\theta\, d\phi, \\ &= \int (C - \cos\phi)\, d\theta \end{aligned}$$

Now, if when $\phi = 0$, $A = 0$,

$$0 = \int (C - 1)\, d\theta,$$

whence $C = 1$; therefore

$$A = \int (1 - \cos\phi)\, d\theta.$$

As an example of the application of this formula, let us take the Loxodrome, the equation to which is

$$\tan \tfrac{1}{2}\phi = \varepsilon^{\theta \cot \alpha},$$

$$\text{or } \log \tan \tfrac{1}{2}\phi = \theta \cot \alpha;$$

$$\text{whence } \frac{d\phi}{\sin\phi} = d\theta \cot\alpha;$$

$$\text{therefore } A = \tan\alpha \int \left(\frac{1 - \cos\phi}{\sin\phi}\right) d\phi,$$

$$= 2 \tan\alpha \int \frac{\sin\frac{\phi}{2}}{\cos\frac{\phi}{2}}\, d\,\frac{\phi}{2};$$

$$\text{therefore } A = C - 2\tan\alpha \log\left(\cos\frac{\phi}{2}\right),$$

$$\text{when } \phi = 0, \quad \cos\frac{\phi}{2} = 1, \quad \log\left(\cos\frac{\phi}{2}\right) = 0,$$

$$\text{when } \phi = \frac{\pi}{2}, \quad \cos\frac{\phi}{2} = \frac{1}{\sqrt{2}}.$$

Therefore from $\phi = 0$ to $\phi = \frac{\pi}{2}$,

$$A = -\,2\tan\alpha \log\frac{1}{\sqrt{2}},$$

$$= 2\tan\alpha \log\sqrt{2} = \tan\alpha \log 2.$$

18. To find the differential expression for the length of a spherical curve.

Referring the curve to rectangular axes (fig. 5.), let $CM = \phi$, $PM = \psi$, and let PP', an element of the curve, $= d\sigma$. Then, as

$$PP'^2 = Pp^2 + P'p^2 \text{ ultimately},$$

$$d\sigma^2 = (\cos\psi)^2\, d\phi^2 + d\psi^2;$$

$$\text{therefore } \frac{d\sigma}{d\phi} = \sqrt{(\cos\psi)^2 + \left(\frac{d\psi}{d\phi}\right)^2}.$$

If we suppose the curve referred to polar coordinates, it is easily seen that

$$\frac{d\sigma}{d\theta} = \sqrt{(\sin\phi)^2 + \left(\frac{d\phi}{d\theta}\right)^2}.$$

If the equation to the curve be

$$\phi = m\theta, \quad \frac{d\phi}{d\theta} = m, \quad \text{and} \quad \frac{d\sigma}{d\theta} = \sqrt{m^2 + (\sin m\theta)^2},$$

which can only be integrated by elliptic functions.

If the curve be the Loxodrome, the equation to which is

$$\tan\frac{\phi}{2} = \varepsilon^{\theta \cot a},$$

$$\frac{d\phi}{\sin\phi} = d\theta \cot a,$$

$$\text{therefore } d\theta^2 = d\phi^2 \frac{(\tan a)^2}{(\sin\phi)^2}.$$

$$\text{Whence, } d\sigma = d\phi \sec a,$$

$$\text{and } \sigma = \phi \sec a + \text{C};$$

which, if taken from $\phi = 0$ to $\phi = \pi$, corresponding to $\theta = -\infty$ and $\theta = \infty$, gives

$$\sigma = \pi \sec a.$$

19. To find the volume of the solid contained between the surface of the sphere and the cylindrical surface passing through any curve of the sphere, and perpendicular to the plane of the equator.

Let PM (fig. 7.) be an elemental prism of the solid, ECM $= \phi$, PCM $= \psi$, $r =$ radius of the sphere. Then

$$\text{height of prism} = r \sin\psi,$$

$$\text{base of prism} = -r\cos\psi\, d\phi \,.\, d(r\cos\psi),$$

(the negative sign being taken as the solid is measured from the surface of the sphere).

$$\begin{aligned}\text{Therefore content of prism} &= -r^2 \sin\psi \cos\psi\, d\phi\, d(r\cos\psi),\\ &= r^3 \sin^2\psi \cos\psi\, d\psi\, d\phi;\\ \text{therefore solid} &= r^3 \iint \sin^2\psi\cos\psi\, d\psi\, d\phi,\\ &= \tfrac{1}{3} r^3 \int (\sin\psi)^3\, d\phi.\end{aligned}$$

As an example, take the curve formed by the intersection of a circular cylinder passing through the centre of the sphere, its equation being $\phi = \psi$,

$$\begin{aligned}\text{solid} &= \tfrac{1}{3} r^3 \int (\sin\phi)^3\, d\phi,\\ &= \tfrac{1}{3} r^3 \int d\phi\, \{1 - (\cos\phi)^2\} \sin\phi,\\ &= \tfrac{1}{3} r^3 \{\cos\phi - \tfrac{1}{3}(\cos\phi)^3\}\, \text{C};\end{aligned}$$

and taking this from $\phi = 0$ to $\phi = \frac{\pi}{2}$,

$$\text{solid} = \tfrac{2}{9} r^3.$$

20. We shall now consider problems connected with the tangencies of spherical curves; and in the first place, as in plane

curves we find the condition for a straight line being a tangent, so we shall investigate the equation to a great circle which touches a spherical curve.

Let the coordinates of the point of contact be ϕ_1, ψ_1: then by (1) the equation to a great circle passing through it is

$$\tan\psi \sin(\phi_1 - a) = \tan\psi_1 \sin(\phi - a).$$

Now, as the circle is to touch the curve, this equation must hold for the point $\phi_1 + d\phi_1$, $\psi_1 + d\psi_1$. Hence, taking the logarithmic differential, we have

$$\frac{\cos(\phi_1 - a)}{\sin(\phi_1 - a)} d\phi_1 = \frac{(\sec\psi_1)^2}{\tan\psi_1} d\psi_1,$$

$$\text{and therefore } \cot(\phi_1 - a) = \frac{(\sec\psi_1)^2}{\tan\psi_1} \frac{d\psi_1}{d\phi_1}.$$

But as $\sin(\phi - a) = \sin(\phi - \phi_1 + \phi_1 - a)$

$$= \sin(\phi - \phi_1)\cos(\phi_1 - a) + \cos(\phi - \phi_1)\sin(\phi_1 - a),$$

$$\tan\psi = \tan\psi_1 \{\sin(\phi - \phi_1)\cot(\phi_1 - a) + \cos(\phi - \phi_1)\}.$$

Whence, eliminating $\cot(\phi_1 - a)$,

$$\tan\psi = (\sec\psi_1)^2 \frac{d\psi_1}{d\phi_1} \sin(\phi - \phi_1) + \tan\psi_1 \cos(\phi - \phi_1),$$

which is the required equation.

21. The equation to the normal circle is readily found from the condition, that it shall be perpendicular to the tangent circle. Now, by (5) the equation to a circle passing through a point (ϕ_1, ψ_1), and perpendicular to a circle whose equation is

$$\tan\psi = m \sin(\phi - a),$$

$$\text{is } \tan\psi = -\frac{1}{m\cos(\phi_1 - a)} \{\sin(\phi - \phi_1) - m \tan\psi_1 \cos(\phi - a)\}.$$

In this case,

$$m = \tan \mathrm{MTN} \text{ and } a - \phi_1 = \mathrm{NT}, \text{ (fig. 6)}.$$

Now, by Napier's rules in the triangle MNT,

$$\sin \mathrm{NT} = \tan\psi_1 \cot \mathrm{MTN};$$

$$\text{therefore } \frac{1}{m} = -\frac{\sin(\phi_1 - a)}{\tan\psi_1},$$

$$\text{and } \frac{1}{m\cos(\phi_1 - a)} = -\frac{\tan(\phi_1 - a)}{\tan\psi_1} = (\cos\psi_1)^2 \frac{d\phi_1}{d\psi_1}.$$

Also, as $\cos(\phi - a) = \cos(\phi - \phi_1 + \phi_1 - a)$,

$$\frac{\cos(\phi - a)}{\cos(\phi_1 - a)} = \cos(\phi - \phi_1) - \tan(\phi_1 - a)\sin(\phi - \phi_1)$$

$$= \cos(\phi - \phi_1) - \sin\psi_1 \cos\psi_1 \sin(\phi - \phi_1) \frac{d\phi_1}{d\psi_1}.$$

Substituting these values, we find

$$\tan\psi = -\frac{d\phi_1}{d\psi_1}\sin(\phi-\phi_1) + \tan\psi_1\cos(\phi-\phi_1),$$

which is the required equation.

22. From these equations, with the aid of Napier's rules, we can easily determine the values of the various parts connected with the tangent and normal.

Let QMQ′ (fig. 6.) be the curve, MT the tangent circle, MK the normal circle, at the point M (φ_1, ψ_1). Then

$$CM = \phi_1, \quad MN = \psi_1.$$

If in the equation to the tangent we make $\psi = 0$, we have

$$\tan(\phi-\phi_1) = \tan NT = -\sin\psi_1\cos\psi_1\frac{d\phi_1}{d\psi_1},$$

which determines NT, and therefore CT.

If we make $\varphi = 0$, we have

$$\tan\psi = \tan CT' = \tan\psi_1\cos\phi_1 - (\sec\psi_1)^2\sin\phi_1\frac{d\psi_1}{d\phi_1}.$$

In the triangle MNT we have, by Napier's rules,

$$\sin MN = \tan NT\cot NMT;$$

$$\text{therefore } \tan NMT = -\cos\psi_1\frac{d\phi_1}{d\psi_1}.$$

In the equation to the normal, if we make $\psi = 0$, we have

$$\tan(\phi-\phi_1) = -\tan NK = \tan\psi_1\frac{d\psi_1}{d\phi_1}.$$

23. If we refer the curve to polar coordinates, P being the pole, and if we put $PM = \phi_1$, $CPM = \theta_1$, we shall easily find the following values of the different parts of the figure, PLM being drawn perpendicular to the tangent and XPS to the radius vector.

$$\tan LPM = \frac{1}{\sin\varphi_1\cos\varphi_1}\frac{d\phi_1}{d\theta_1},$$

$$\tan PS = \sin^2\phi_1\frac{d\theta_1}{d\phi_1}, \quad \sin PL = \frac{\sin^2\phi_1}{\sqrt{\sin^2\phi_1 + \left(\frac{d\phi_1}{d\theta_1}\right)^2}}$$

$$\tan ML = \frac{\tan\phi_1}{\sqrt{\sin^2\phi_1 + \left(\frac{d\phi_1}{d\theta_1}\right)^2}},$$

$$\cos MX = \frac{\cos\phi_1}{\sqrt{1 + \left(\frac{d\phi_1}{d\theta_1}\right)^2}}, \quad \tan PX = \frac{d\phi_1}{d\theta_1},$$

24. Having discussed the principal points of the theory of tangents, we shall now consider the curvature of spherical curves. This, as in plane curves, will be determined by the radius of the small circle of the sphere which has a contact of the second order with the given curve, so that the principle of the investigation is the same as that in plane curves, though the forms of the equations make the expressions much more complicated. The equation to a small circle whose radius is γ, and the coordinates of the pole of which are a, β, is by (4),

$$\cos\gamma = \cos\beta\cos\psi\cos(\phi - a) + \sin\beta\sin\psi \ldots\ldots (1).$$

Differentiating this, considering ϕ and ψ as variable, we have

$$0 = \cos\beta\,\{\sin\psi\cos(\phi - a)\,d\psi + \cos\psi\sin(\phi - a)\,d\phi\} - \sin\beta\cos\psi\,d\psi;$$

whence $$\tan\beta = \tan\psi\cos(\phi - a) + \sin(\phi - a)\frac{d\phi}{d\psi} \ldots\ldots (2).$$

Differentiating a second time,

$$0 = \sec^2\psi\cos(\phi - a) - \tan\psi\sin(\phi - a)\frac{d\phi}{d\psi} + \cos(\phi - a)\left(\frac{d\phi}{d\psi}\right)^2 + \sin(\phi - a)\frac{d^2\phi}{d\psi^2},$$

which may be put under the form

$$A\cos(\phi - a) - B\sin(\phi - a) = 0 \ldots\ldots (3),$$

$$\text{where } A = (\sec\psi)^2 + \left(\frac{d\phi}{d\psi}\right)^2,$$

$$\text{and } B = \tan\psi\frac{d\phi}{d\psi} - \frac{d^2\phi}{d\psi^2}.$$

Now, from (3) we have

$$\tan(\phi - a) = \frac{A}{B},$$

$$\text{therefore } \cos(\phi - a) = \frac{B}{\sqrt{A^2 + B^2}},$$

$$\text{and } \sin(\phi - a) = \frac{A}{\sqrt{A^2 + B^2}}.$$

Substituting these in (2), we find

$$\tan\beta = \frac{B\tan\psi + A\dfrac{d\phi}{d\psi}}{\sqrt{A^2 + B^2}} \ldots\ldots\ldots (4).$$

Also, equation (3) may be put under the form

$$A(\cos\phi\cos a + \sin\phi\sin a) - B(\sin\phi\cos a - \cos\phi\sin a) = 0;$$

whence

$$\tan a = -\frac{A\cos\phi - B\sin\phi}{A\sin\phi + B\cos\phi} \;\ldots\ldots\; (5):$$

and thus the coordinates of the centre of curvature are determined. To find the value of γ, we have from (4)

$$\sin\beta = \frac{B\tan\psi + A\dfrac{d\phi}{d\psi}}{\sqrt{A^2 + B^2 + \left(B\tan\psi + A\dfrac{d\phi}{d\psi}\right)^2}}$$

$$\text{and}\;\cos\beta = \frac{\sqrt{A^2 + B^2}}{\sqrt{A^2 + B^2 + \left(B\tan\psi + A\dfrac{d\phi}{d\psi}\right)^2}}.$$

Substituting these values, and that of $\cos(\phi - a)$ in (1), we have

$$\cos\gamma = \frac{B\cos\psi + \left(B\tan\psi + A\dfrac{d\phi}{d\psi}\right)\sin\psi}{\sqrt{A^2 + B^2 + \left(B\tan\psi + A\dfrac{d\phi}{d\psi}\right)^2}}$$

$$= \frac{B + A\sin\psi\cos\psi\dfrac{d\phi}{d\psi}}{\sqrt{A^2\cos^2\psi\left\{1 + \left(\dfrac{d\phi}{d\psi}\right)^2\right\} + 2AB\sin\psi\cos\psi + B^2}} \;\ldots(6).$$

25. As an example of the application of these formulæ, let us take the curve, the equation to which is

$$\phi = \psi;$$

$$\text{whence}\;\frac{d\phi}{d\psi} = 1,\quad \frac{d^2\phi}{d\psi^2} = 0.$$

This gives

$$A = 1 + (\sec\psi)^2,\qquad B = \tan\psi;$$

and substituting these in (4), we find

$$\tan\beta = \frac{2}{\sqrt{1 + 3(\cos\phi)^2}},\quad \text{since}\;\phi = \psi.$$

Also, substituting these values of A and B in (5), and observing that from the equation to the curve $\phi = \psi$, we obtain, after some reductions,

$$\tan a = -\frac{2\cot\phi}{2 + (\sec\phi)^2},$$

$$\text{or}\;\cot a = -\tfrac{1}{2}\tan\phi\,\{3 + (\tan\phi)^2\}.$$

Lastly, substituting the same values in (6) after reducing, we obtain

$$\cos\gamma = \frac{\sin\phi\,\{2 + (\cos\phi)^2\}}{\sqrt{5 + 3(\cos\phi)^2}};$$

$$\text{when } \phi = 0, \quad \gamma = \frac{\pi}{2}; \quad \text{when } \phi = \frac{\pi}{2}, \quad \cos\gamma = \frac{2}{\sqrt{5}}.$$

26. The equation to the evolute of a spherical curve will be found in the same way as that of a plane curve, by eliminating ϕ and ψ between (4) and (5). If we take as an example the same curve as before,

$$\phi = \psi,$$

we have
$$4(\cot\beta)^2 = 1 + 3(\cos\phi)^2,$$
$$\text{and } 4(\cot\alpha)^2 = \tan^2\phi\,\{3 + (\tan\phi)^2\}^2;$$
$$\text{therefore } (\cos\phi)^2 = \frac{4(\cot\beta)^2 - 1}{3},$$
$$\text{and } (\tan\phi)^2 = \frac{4\{1 - (\cot\beta)^2\}}{4(\cot\beta)^2 - 1}.$$

Substituting this value,

$$(\cot\alpha)^2 = \frac{\{1 - (\cot\beta)^2\}\{8(\cot\beta)^2 - 1\}^2}{\{4(\cot\beta)^2 - 1\}^3},$$
$$\text{or } (\tan\alpha)^2 = \frac{\{4 - (\tan\beta)^2\}^3}{\{(\tan\beta)^2 - 1\}\{8 - (\tan\beta)^2\}^2},$$

which is the equation to the evolute.

27. As another example, let us take the Loxodrome. This curve is best referred to polar coordinates, and we must therefore adapt our formulæ to the change, which is easily done by putting for ψ and β, $\frac{\pi}{2} - \psi$ and $\frac{\pi}{2} - \beta$, and leaving ϕ and α unchanged. The equation to the Loxodrome is

$$\tan\frac{\psi}{2} = \varepsilon^{\phi\cot c};$$

$$\text{whence } \frac{d\psi}{\sin\psi} = d\phi\cot c;$$

which gives us

$$\mathrm{A} = \frac{\{1 + (\tan c)^2\}}{(\sin\psi)^2} = \frac{(\sec c)^2}{(\sin\psi)^2}, \quad \mathrm{B} = 0.$$

$$\text{whence } \cot\beta = \frac{\tan c}{\sin\psi},$$

$$\tan\alpha = -\cot\phi, \text{ or } \alpha = \frac{\pi}{2} + \phi,$$

$$\text{and } \cos\gamma = \frac{\tan c \sin\psi}{\sqrt{(\sin\psi)^2 + (\tan c)^2}},$$

which determines the radius of curvature.

For the evolute we have

$$\tan\beta \tan c = \sin\psi = \frac{2\tan\frac{\psi}{2}}{1 + \left(\tan\frac{\psi}{2}\right)^2}.$$

Putting for $\tan\frac{\psi}{2}$ its value from the equation to the curve, and putting $a - \frac{\pi}{2}$ for ϕ, we get

$$\tan\beta \tan c = \frac{2\epsilon^{\left(a - \frac{\pi}{2}\right)\cot c}}{1 + \epsilon^{2\left(a - \frac{\pi}{2}\right)\cot c}};$$

or dividing the numerator and denominator by $\epsilon^{\left(a - \frac{\pi}{2}\right)\cot c}$, and inverting,

$$\cot\beta \cot c = \tfrac{1}{2}\left\{\epsilon^{\left(a - \frac{\pi}{2}\right)\cot c} + \epsilon^{-\left(a - \frac{\pi}{2}\right)\cot c}\right\},$$

which is the equation to the evolute.

S. S. G.

X.—ON A PROPERTY OF THE BRACHYSTOCHRONE WHEN THE FORCES ARE ANY WHATEVER.

Sir,—Perhaps you may think fit for insertion the following simple proof, and extension to curves of double curvature, of a Property of the Brachystochrone, demonstrated in Mr. Whewell's *Dynamics*, Part II. sect 4.

The curve on which a material point will move from one fixed point to another (or from a point to a curve, &c.) in the shortest time under the action of known forces, is such, that the parts of the pressure on it arising from those forces and from the centrifugal force are equal.

Since $t = \int \frac{ds}{v}$, we must have $\delta t = \int \frac{v\delta ds - ds\,\delta v}{v^2} = 0$ between the given limits.

$$\text{Now, } ds^2 = dx^2 + dy^2 + dz^2;$$

$$\text{therefore}\quad \frac{ds}{dt}\,\delta ds = \frac{dx}{dt}\,\delta dx + \frac{dy}{dt}\,\delta dy + \frac{dz}{dt}\,\delta dz\,;$$

and if XYZ be the components of the accelerating forces acting on the point, then

$$v^2 = \text{const.} + 2\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz).$$

$$\text{Hence}\quad v\delta v = \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z,$$

$$\text{and}\quad ds\,\delta v = (\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z)\,dt\,;$$

on substituting these values, we have

$$\delta t = \int \frac{1}{v^2}\left(\frac{dx}{dt}\,d\delta x + \frac{dy}{dt}\,d\delta y + \frac{dz}{dt}\,d\delta z\right) - \int dt\,\frac{1}{v^2}\,(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z) = 0,$$

$$\text{or}\quad \delta t = \frac{1}{v^2}\left(\frac{dx}{dt}\,\delta x + \frac{dy}{dt}\,\delta y + \frac{dz}{dt}\,\delta z\right)$$

$$-\int dt\left\{\frac{d}{dt}\left(\frac{1}{v^2}\frac{dx}{dt}\right)\delta x + \frac{d}{dt}\left(\frac{1}{v^2}\frac{dy}{dt}\right)\delta y + \frac{d}{dt}\left(\frac{1}{v^2}\frac{dz}{dt}\right)dz\right\}$$

$$-\int dt\,\frac{1}{v^2}(\mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z) = 0,$$

by integrating the first term by parts.

Now, in order that the part under the sign of integration may vanish, the coefficients of the variations must separately equal zero, there being no connexion between them. Hence

$$\frac{d}{dt}\left(\frac{1}{v^2}\frac{dx}{dt}\right) + \frac{\mathrm{X}}{v^2} = 0,\quad \frac{d}{dt}\left(\frac{1}{v^2}\frac{dy}{dt}\right) + \frac{\mathrm{Y}}{v^2} = 0,\quad \frac{d}{dt}\left(\frac{1}{v^2}\frac{dz}{dt}\right) + \frac{\mathrm{Z}}{v^2} = 0\,;$$

or changing the independent variable,

$$v^2\frac{d^2x}{ds^2} - \frac{dx}{ds}\,v\,\frac{dv}{ds} + \mathrm{X} = 0,\quad v^2\frac{d^2y}{ds^2} - \frac{dy}{ds}\,v\,\frac{dv}{ds} + \mathrm{Y} = 0,$$

$$v^2\frac{d^2z}{ds^2} - \frac{dz}{ds}\,v\,\frac{dv}{ds} + \mathrm{Z} = 0.$$

If we now multiply these equations by

$$\frac{d^2x}{ds^2},\quad \frac{d^2y}{ds^2},\quad \frac{d^2z}{ds^2},$$

and add the results, (remembering that since

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1,\qquad \frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0,)$$

we find

$$\frac{v^2}{\rho} = -\left(\mathrm{X}\rho\,\frac{d^2x}{ds^2} + \mathrm{Y}\rho\,\frac{d^2y}{ds^2} + \mathrm{Z}\rho\,\frac{d^2z}{ds^2}\right),$$

ρ being the radius of curvature of the curve at the point xyz.

This result expresses the property above enunciated: for the equations of motion of the point may be put in the form

$$\frac{ds^2}{dt^2}\frac{d^2x}{ds^2} + \frac{dx}{ds}\frac{d^2s}{dt^2} = \mathrm{X} + n\rho\,\frac{d^2x}{ds^2},\quad \&\text{c.}$$

where n is the pressure on the curve, or its reaction, which acts in the direction of the radius of curvature.

If we multiply by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ respectively, and add, we get the expression above used for the velocity; but if we multiply them by $\rho\frac{d^2x}{ds^2}$, $\rho\frac{d^2y}{ds^2}$, $\rho\frac{d^2z}{ds^2}$, and add, we find

$$\mathrm{R} = \frac{v^2}{\rho} - \left(\mathrm{X}\rho\frac{d^2x}{ds^2} + \mathrm{Y}\rho\frac{d^2z}{ds^2} + \mathrm{Z}\rho\frac{d^2z}{ds^2}\right).$$

If the curve is restricted to be drawn on a curve surface, whose equation is $\mathrm{L} = 0$, the variations are connected by the equation

$$\frac{d\mathrm{L}}{dx}\delta x + \frac{d\mathrm{L}}{dy}\delta y + \frac{d\mathrm{L}}{dz}\delta z = 0,$$

and we get two equations, from which it is impossible to deduce the property analogous to the above.

If the curve lie in one plane (as that of xy), and the only force act parallel to the axis of x, then the second equation of condition becomes $\frac{1}{v}\frac{dy}{ds} = \text{const.}$ If the force be that of gravity then

$$v = \sqrt{2gx}, \text{ and } \frac{dy}{ds} = \text{const.} \sqrt{2gx} = \sqrt{\frac{x}{2a}} \text{ suppose:}$$

$$\text{therefore } \frac{dx^2}{dy^2} = \frac{ds^2}{dy^2} - 1 = \frac{2a - x}{x},$$

$$\text{and } \frac{dy}{dx} = \frac{x}{\sqrt{2ax - x^2}} = \frac{a}{\sqrt{2ax - x^2}} - \frac{a - x}{\sqrt{2ax - x^2}};$$

$$\text{therefore } y = a \text{ vers.}^{-1}\frac{x}{a} - \sqrt{2ax - x^2},$$

and the brachystochrone is in this case a common cycloid.

I remain, Sir, yours, &c.

Oxford, Oct. 17, 1839. D.

XI.—MATHEMATICAL NOTES.

1. Given the n^{th} part of a straight line, to find the $(n + 1)^{\text{th}}$ part.

Let AP (fig. 8.) be the n^{th} part of AB. Upon AB describe a square, and draw the diagonal AD; join PC, and through E draw FEQ parallel to AC or BD. AQ will be the $(n + 1)^{\text{th}}$ part of

AB. For by similar triangles EQ is to EF as AP to CD. But AP is the n^{th} part of CD; therefore EQ is the n^{th} part of EF, *i.e.* (since AQ = EQ and EF = FD = QB), AQ is the n^{th} part of QB, and therefore the $(n+1)^{\text{th}}$ part of AB. So by joining QC we may find the $(n+2)^{\text{th}}$ part, and so on successively.

This very simple problem is given by Meibomius in the Preface to his edition of *Aristides Quintilianus,* having been suggested, he says, by a figure called Helicon, and used by the ancient writers on harmonics.

2. *Property of the Parabola.*—In Vol. I. p. 205, there were found for the coordinates of the point of intersection of two tangents to a parabola, the expressions

$$y = m\,(a + a'), \quad x = m\,a\,a',$$

a, a' being the tangents of the angles which the tangents to the curve make with the axis of y. From these expressions it follows, that if y_1, y_2, &c. x_1, x_2, &c. be the coordinates of the angles of any re-entering polygon of $2n$ sides circumscribing a parabola,

$$y_1 - y_2 + y_3, \text{ \&c. } - y_{2n} = 0,$$

$$\text{and } \frac{x_1 x_3 \;.\;.\;.\;.\; x_{2n-1}}{x_2 x_n \;.\;.\;.\;.\; x_{2n}} = 1.$$

Also, the continued product of the abscissæ of the points of intersection of any number of tangents, is equal to the continued product of the abscissæ of the points of contact, provided no three points of intersection lie in the same straight line.

Let x', x'', x''', &c. be the abscissæ of the points of contact, then it is easily seen, from the equation to the parabola, that

$$x' = ma'^2, \quad x'' = ma''^2, \quad x''' = ma'''^2, \text{ \&c.}$$

the continued product of which is

$$x'x''x''' \;.\;.\;.\; x^{(n)} = m^n a'^2 a''^2 a'''^2 \;.\;.\;.\; a^{(n)2}.$$

And if x_1, x_2, x_3, &c. be the coordinates of the points of intersection of the tangents, we have

$$x_1 = ma'a'', \quad x_2 = ma''a''', \quad x_3 = ma''a''', \text{ \&c.}$$

the continued product of which is

$$x_1 x_2 x_3 \;.\;.\;.\; x_n = m^n \cdot a'^2\, a''^2\, a'''^2 \;.\;.\;.\; a^{(n)\,2},$$

which is equal to the preceding expression. It is necessary to limit the intersections in such a way that no three shall lie in the same line, because otherwise some one of the a's in the second series would appear more than twice.

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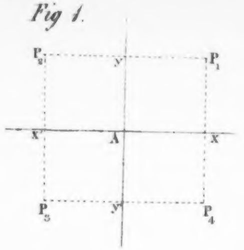
Fig 1.



Fig

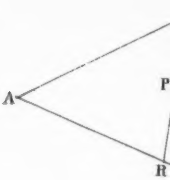


Fig 3.

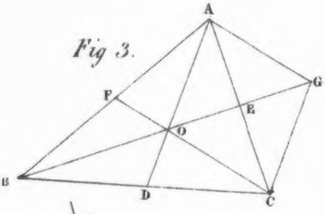
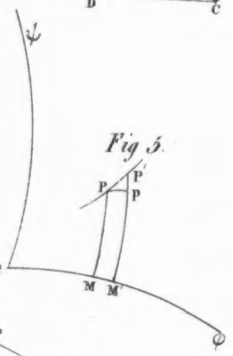


Fig 5.



Fig

Fig 6.

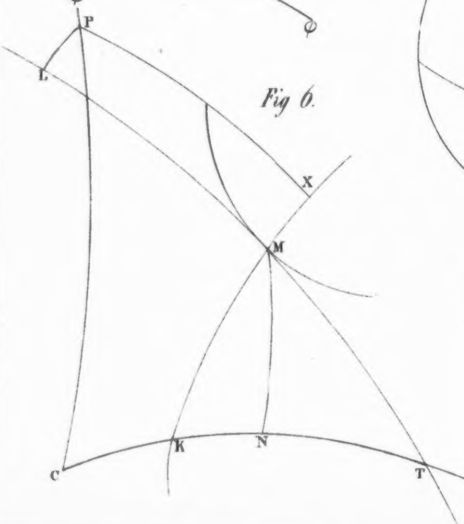


PLATE I.

Fig 2.

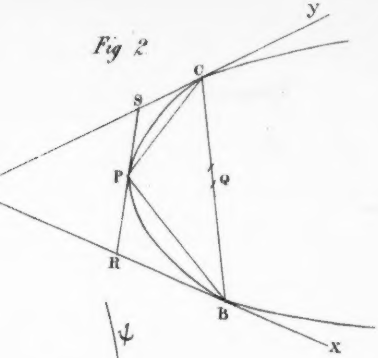


Fig 4.

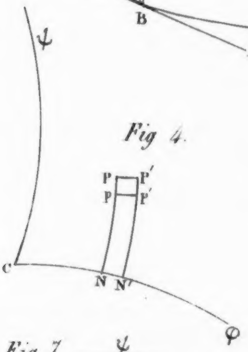


Fig 7.

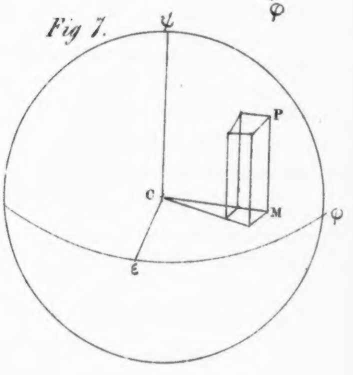
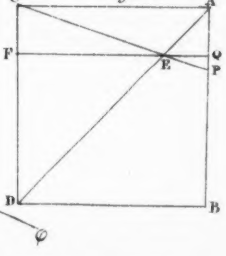


Fig 8.



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